QUANTIZED REDUCTIONS AND IRREDUCIBLE
REPRESENTATIONS OF $\mathcal{W}$-ALGEBRAS

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Dedicated to Akihiro Tsuchiya on the occasion of his 60th birthday

ABSTRACT. We study the representations of the $\mathcal{W}$-algebra $\mathcal{W}(\bar{\mathfrak{g}})$ associated to an arbitrary finite-dimensional simple Lie algebra $\bar{\mathfrak{g}}$ via the quantized Drinfeld-Sokolov reductions. The characters of irreducible representations of $\mathcal{W}(\bar{\mathfrak{g}})$ with non-degenerate highest weights are expressed by Kazhdan-Lusztig polynomials. The irreducibility conjecture of Frenkel, Kac and Wakimoto is proved completely for the "−" reduction and partially for the "+" reduction. In particular, the existence of the minimal series representations (= the modular invariant representations) of $\mathcal{W}(\bar{\mathfrak{g}})$ is proved.

1. INTRODUCTION

Since introduced by Zamalodchikov [23], the symmetry by $\mathcal{W}$-algebras has been significantly important in conformal field theories ([2, 3]). However, not much is known about the representation theory of $\mathcal{W}$-algebras. In this paper we study the representations of the $\mathcal{W}$-algebra $\mathcal{W}(\bar{\mathfrak{g}})$ associated to an arbitrary simple Lie algebra $\bar{\mathfrak{g}}$, and determine the characters of its irreducible highest weight representations under certain conditions. Those characters are expressed by Kazhdan-Lusztig polynomials. As a consequence, we prove the conjecture of Frenkel, Kac and Wakimoto ([11]) on the existence of the minimal series representations (= the modular invariant representations) of $\mathcal{W}(\bar{\mathfrak{g}})$.

Let $\bar{\mathfrak{g}}$ be a finite-dimensional simple Lie algebra with a triangular decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_+$. Let $\mathfrak{g} = \bar{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}\mathbf{D}$ be the affine Lie algebra associated to $\bar{\mathfrak{g}}$. Let $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$ be the $\mathcal{W}$-algebra associated to $\bar{\mathfrak{g}}$ at level $\kappa - h^\vee$, defined by Feigin and Frenkel via the quantized Drinfeld-Sokolov reduction [8, 10]. We have:

$$\mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}})) \cong \mathcal{Z}(\bar{\mathfrak{g}}), \tag{1}$$

where $\mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$ is the Zhu algebra of $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$ ([12]) and $\mathcal{Z}(\bar{\mathfrak{g}})$ is the center of the universal enveloping algebra $U(\bar{\mathfrak{g}})$, see Theorem 4.2.3. By (1), the irreducible highest weight representations of $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$ are parameterized by central characters of $\mathcal{Z}(\bar{\mathfrak{g}})$. Let $\gamma_{\bar{\lambda}} : \mathcal{Z}(\bar{\mathfrak{g}}) \to \mathbb{C}$ be the central character defined as the evaluation at the Verma module of $\bar{\mathfrak{g}}$ of highest weight $\bar{\lambda} \in \bar{\mathfrak{h}}^*$. Let $\mathbf{L}(\gamma_{\bar{\lambda}})$ be the irreducible representation of $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$ of highest weight $\gamma_{\bar{\lambda}}$, that is, the irreducible representation $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$ corresponding to $\gamma_{\bar{\lambda}}$. To determine the characters of all $\mathbf{L}(\gamma_{\bar{\lambda}})$ is, certainly, one of the most important problems in representation theory of $\mathcal{W}$-algebras.

Let $\mathcal{O}_\kappa$ be the Bernstein-Gelfand-Gelfand category of $\mathfrak{g}$ of level $\kappa - h^\vee$. Let $H_+^\bullet(V)$, $V \in \mathcal{O}_\kappa$, be the cohomology associated to the quantized Drinfeld-Sokolov reduction defined by Feigin and Frenkel ([8]). Thus, $H_+^\bullet(V) = H^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{n}}_+, V \otimes \mathbb{C}_{\chi_+})$, where $L\bar{\mathfrak{n}}_+ = \bar{\mathfrak{n}}_+ \otimes \mathbb{C}[t, t^{-1}] \subset \mathfrak{g}$ and $\mathbb{C}_{\chi_+}$ is a certain one-dimensional representation of $L\bar{\mathfrak{n}}_+$. Then, the correspondence $V \rightsquigarrow H_+^i(V)$ ($i \in \mathbb{Z}$) defines a family of functors from $\mathcal{O}_\kappa$ to the category of $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$-modules ([8]). However, because the description of this functor is rather complicated in general, Frenkel, Kac and Wakimoto [11] introduced another functor $V \rightsquigarrow H_-^i(V)$ ($i \in \mathbb{Z}$) from $\mathcal{O}_\kappa$ to the category of $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$-modules, defined by $H_-^\bullet(V) = H^{\frac{\infty}{2}+\bullet}(L\bar{\mathfrak{n}}_-, V \otimes \mathbb{C}_{\chi_-})$. Here, $L\bar{\mathfrak{n}}_- = \bar{\mathfrak{n}}_- \otimes \mathbb{C}[t, t^{-1}] \subset \mathfrak{g}$ and $\mathbb{C}_{\chi_-}$ is again a certain one-dimensional representation of $L\bar{\mathfrak{n}}_-$. It turns out that the corresponding functor

$$V \rightsquigarrow H_-^0(V)$$

indeed has nice properties: it is exact, it sends Verma modules to Verma modules, and simple modules to simple modules under certain conditions.

We now describe our results more precisely. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ and let $\mathfrak{h}^* \ni \Lambda \mapsto \bar{\Lambda} \in \bar{\mathfrak{h}}^*$ be the restriction. Let $M(\Lambda)$ be the Verma module of $\mathfrak{g}$ of highest weight $\Lambda \in \mathfrak{h}^*$, $L(\Lambda)$ its unique simple quotient. Let $\bar{\Delta}_+$ be the set of positive roots of $\bar{\mathfrak{g}}$, identified with the subset of the set Δ_+^{re} of positive real roots of $\mathfrak{g}$ in the standard way.

Main Theorem 1 (Theorem 6.5.1)**.** *Let Λ be non-degenerate (that is, $\langle \Lambda, \bar{\alpha}^\vee \rangle \notin \mathbb{Z}$ for all $\bar{\alpha} \in \bar{\Delta}_+$) and non-critical (that is, $\kappa = \langle \Lambda + \rho, K \rangle \neq 0$). Then, $H_-^0(L(\Lambda)) \cong \mathbf{L}(\gamma_{\bar{\Lambda}})$.*

Note that the condition that Λ is non-degenerate does not imply that the integral Weyl group W^Λ of Λ is finite (and indeed it is infinite when Λ is admissible, see [16]). We also show that $H_-^0(M(\lambda)) \cong \mathbf{M}(\gamma_{\bar{\lambda}})$, where $\mathbf{M}(\gamma_{\bar{\lambda}})$ is the Verma module of $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$ of highest weight $\gamma_{\bar{\lambda}}$, see Theorem 6.1.2. Thus, combined with our previous result [1], Main Theorem 1 gives the character of $\mathbf{L}(\gamma_{\bar{\lambda}})$ with any non-degenerate highest weight $\gamma_{\bar{\lambda}}$, see Theorem 6.6.1. Here, an infinitesimal character $\gamma_{\bar{\lambda}}$ ($\bar{\lambda} \in \bar{\mathfrak{h}}^*$) is called non-degenerate if $\langle \bar{\lambda}, \bar{\alpha}^\vee \rangle \notin \mathbb{Z}$ for all $\bar{\alpha} \in \bar{\Delta}_+$.

By applying Main Theorem 1 to non-degenerate admissible weights ([16]), the irreduciblity conjecture of Frenkel, Kac and Wakimoto [11] is proved (for the "−"-reduction). Therefore, combined with our previous result [1], the existence of modular invariant representations of $\mathcal{W}(\bar{\mathfrak{g}})$ is proved.

Now the "+"-reduction is not as simple as the "−"-reduction. However, we have the following theorem.

Main Theorem 2 (Theorem 6.7.1)**.** *Suppose that $\Lambda \in \mathfrak{h}^*$ is non-critical and satisfies the following condition:*

$$\langle \Lambda, \alpha^\vee \rangle \notin \mathbb{Z} \quad \text{for all } \alpha \in \{-\bar{\alpha} + n\delta; \bar{\alpha} \in \bar{\Delta}_+, 1 \leq n \leq \mathrm{ht}\, \alpha\}.$$

Then, $H_+^0(L(\Lambda)) \cong \mathbf{L}(\gamma_{\overline{t_{-\bar{\rho}^\vee} \circ \Lambda}})$.

This paper is organized as follows. In Section 2, we collect the necessary information about the affine Lie algebra $\mathfrak{g}$ and the BRST complexes associated to quantized Drinfeld-Sokolov reductions. In Section 3, we recall the definition of $\mathcal{W}$-algebra $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$ and the collect necessary information about its structure. In Section 4, we prove that the Zhu algebra $\mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$ of $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$ is canonically isomorphic to

the center $\mathcal{Z}(\bar{\mathfrak{g}})$ of $U(\bar{\mathfrak{g}})$ (Theorem 4.2.3). In Section 5, we define Verma modules $\mathbf{M}(\gamma_{\bar{\lambda}})$ of $\mathcal{W}_{\kappa}(\bar{\mathfrak{g}})$ of highest weight $\gamma_{\bar{\lambda}}$ and its simple quotient $\mathbf{L}(\gamma_{\bar{\lambda}})$. The duality structure of modules over $\mathcal{W}$-algebras is also discussed. Finally, in Section 6, we prove Main Theorems.

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2. Preliminaries on the affine Lie algebra and the BRST complex

In this section we collect the necessary information about the affine Lie algebra and the BRST complexes associated to quantized Drinfeld-Sokolov reductions. Our basic references in this section are the textbooks [13, 22, 10].

2.1. **The affine Lie algebra $\mathfrak{g}$.** In the sequel, we fix a simple finite-dimensional complex Lie algebra $\bar{\mathfrak{g}}$ and a Cartan subalgebra $\bar{\mathfrak{h}} \subset \bar{\mathfrak{g}}$. Let $\bar{\Delta}$ denote the set of roots, $\bar{\Pi}$ a basis of $\bar{\Delta}$, $\bar{\Delta}_+$ the set of positive roots, and $\bar{\Delta}_- = -\bar{\Delta}_+$. This gives the triangular decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_+$. Let $\bar{Q}$ denote the root lattice, $\bar{P}$ the weight lattice, $\bar{Q}^\vee$ the coroot lattice and $\bar{P}^\vee$ the coweight lattice. Let $\bar{\rho}$ be the half sum of positive roots, $\bar{\rho}^\vee$ the half sum of positive coroots. For $\alpha \in \bar{\Delta}_+$, the number $\langle \alpha, \bar{\rho}^\vee \rangle$ is called the *height* of α and denote by $\operatorname{ht}\alpha$. Let $\bar{W}$ be the Weyl group of $\bar{\mathfrak{g}}$, w_0 the longest element of $\bar{W}$.

Let $(\, , \,)$ be the normalized invariant inner product of $\bar{\mathfrak{g}}$. Thus, $(\, , \,) = \frac{1}{2h^\vee}$Killing form, where $h^\vee$ is the dual Coxeter number of $\bar{\mathfrak{g}}$. We identify $\bar{\mathfrak{h}}$ and $\bar{\mathfrak{h}}^*$ using the form. Then, $\alpha^\vee = 2\alpha/(\alpha, \alpha)$, $\alpha \in \bar{\Delta}$.

Let $l = \operatorname{rank}\bar{\mathfrak{g}}$, $\bar{I} = \{1, 2, \ldots, l\}$. Choose a basis $\{J_a; a \in \bar{I} \sqcup \bar{\Delta}\}$ of $\bar{\mathfrak{g}}$ such that $J_\alpha \in \bar{\mathfrak{g}}_\alpha$, $(J_\alpha, J_{-\alpha}) = 1$ and $(J_\alpha)^t = J_{-\alpha}$ $(\alpha \in \bar{\Delta})$. Here, $\bar{\mathfrak{g}} \ni X \mapsto X^t \in \bar{\mathfrak{g}}$ is the Chevalley anti-automorphism. Let $c_{a,b}^c$ be the structure constant with respect to this basis; $[J_a, J_b] = \sum_c c_{a,b}^c J_c$. Then, $c_{\alpha,\beta}^\gamma = -c_{-\alpha,-\beta}^{-\gamma}$ $(\alpha, \beta, \gamma \in \bar{\Delta}_+)$.

Let $\mathfrak{g} = \bar{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}\mathbf{D}$ be the affine Lie algebra associated to $(\bar{\mathfrak{g}}, (\, , \,))$, where K is its central element and $\mathbf{D}$ is the degree operator ([13]). The bilinear form $(\, , \,)$ is naturally extended from $\bar{\mathfrak{g}}$ to $\mathfrak{g}$. Set $X(n) = X \otimes t^n$, $X \in \bar{\mathfrak{g}}$, $n \in \mathbb{Z}$. The subalgebra $\bar{\mathfrak{g}} \otimes \mathbb{C} \subset \mathfrak{g}$ is naturally identified with $\bar{\mathfrak{g}}$.

Fix the triangular decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ in the standard way. Thus,

$$\mathfrak{h} = \bar{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}\mathbf{D},$$
$$\mathfrak{g}_- = \bar{\mathfrak{n}}_- \otimes \mathbb{C}[t^{-1}] \oplus \bar{\mathfrak{h}} \otimes \mathbb{C}[t^{-1}]t^{-1} \oplus \bar{\mathfrak{n}}_+ \otimes \mathbb{C}[t^{-1}]t^{-1},$$
$$\mathfrak{g}_+ = \bar{\mathfrak{n}}_- \otimes \mathbb{C}[t]t \oplus \bar{\mathfrak{h}} \otimes \mathbb{C}[t]t \oplus \bar{\mathfrak{n}}_+ \otimes \mathbb{C}[t].$$

Let $\mathfrak{h}^* = \bar{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$ be the dual of $\mathfrak{h}$. Here, Λ_0 and δ are dual elements of K and $\mathbf{D}$ respectively. For $\lambda \in \mathfrak{h}^*$, the number $\langle \lambda, K \rangle$ is called the *level of* λ. Let $\mathfrak{h}_\kappa^*$ denote the set of the weights of level $\kappa - h^\vee$:

$$(2) \qquad \mathfrak{h}_\kappa^* = \{\lambda \in \mathfrak{h}^*; \langle \lambda + \rho, K \rangle = \kappa\},$$

where, $\rho = \bar{\rho} + h^\vee \Lambda_0 \in \mathfrak{h}^*$. Let $\bar{\lambda}$ be the restriction of $\lambda \in \mathfrak{h}^*$ to $\bar{\mathfrak{h}}^*$.

Let Δ be the set of roots of $\mathfrak{g}$, Δ_+ the set of positive roots, $\Delta_- = -\Delta_+$. Then, $\Delta = \Delta^{\mathrm{re}} \sqcup \Delta^{\mathrm{im}}$, where Δ^{re} is the set of real roots and Δ^{im} is the set of imaginary roots. Let Π be the standard basis of Δ^{re}, $\Delta_\pm^{\mathrm{re}} = \Delta^{\mathrm{re}} \cap \Delta_\pm$, $\Delta_\pm^{\mathrm{im}} = \Delta^{\mathrm{im}} \cap \Delta_\pm$. Let Q be the root lattice, $Q_+ = \sum_{\alpha \in \Delta_+} \mathbb{Z}_{\geq 0}\alpha \subset Q$.

Let $W \subset GL(\mathfrak{h}^*)$ be the Weyl group of $\mathfrak{g}$ generated by the reflections s_α, $\alpha \in \Delta^{\mathrm{re}}$, defined by $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$. Then, $W = \bar{W} \ltimes \bar{Q}^\vee$. Let $\widetilde{W} = \bar{W} \ltimes \bar{P}^\vee$, the extended Weyl group of $\mathfrak{g}$. For $\mu \in \bar{P}^\vee$, we denote the corresponding element of $\widetilde{W}$ by t_μ. Then,

$$t_\mu(\lambda) = \lambda + \langle \lambda, K \rangle \mu - \left(\langle \lambda, \mu \rangle + \frac{1}{2} |\mu|^2 \langle \lambda, K \rangle \right) \delta \quad (\lambda \in \mathfrak{h}^*).$$

Let $\widetilde{W}_+ = \{ w \in \widetilde{W}; \Delta_+^{\mathrm{re}} \cap w^{-1}(\Delta_-^{\mathrm{re}}) = \emptyset \}$. We have: $\widetilde{W} = \widetilde{W}_+ \ltimes W$. The dot action of $\widetilde{W}$ on $\mathfrak{h}^*$ is defined by $w \circ \lambda = w(\lambda + \rho) - \rho$ $(\lambda \in \mathfrak{h}^*)$. We have the natural homomorphism $\widetilde{W} \to \mathrm{Aut}(\mathfrak{g})$ such that $w(\mathfrak{g}_\alpha) \subset \mathfrak{g}_{w(\alpha)}$ $(w \in \widetilde{W}, \alpha \in \Delta)$.

For $\Lambda \in \mathfrak{h}^*$, let

$$(3) \qquad\qquad R^\Lambda = \{ \alpha \in \Delta^{\mathrm{re}}; \langle \Lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z} \},$$

$R_+^\Lambda = R^\Lambda \cap \Delta_+^{\mathrm{re}}$. It is known that R^Λ is a subroot system of Δ^{re} ([22, 17]). Let

$$(4) \qquad\qquad W^\Lambda = \langle s_\alpha; \alpha \in R^\Lambda \rangle \subset W.$$

The Coxeter group W^Λ is called the *integral Weyl group* of Λ. We have:

$$(5) \qquad\qquad R^{w \circ \Lambda} = R^\Lambda \text{ for all } w \in W^\Lambda.$$

2.2. **The BGG category of $\mathfrak{g}$.** For a $\mathfrak{g}$-module V (or for simply an $\mathfrak{h}$-module V), let $V^\lambda = \{ v \in V; hv = \lambda(h)v \text{ for } h \in \mathfrak{h} \}$ be the weight space of weight λ. Let $P(V) = \{ \lambda \in \mathfrak{h}^*; V^\lambda \neq \{0\} \}$. If $\dim V^\lambda < \infty$ for all λ, then we set

$$(6) \qquad\qquad V^* = \bigoplus_\lambda \mathrm{Hom}_{\mathbb{C}}(V^\lambda, \mathbb{C}) \subset \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}).$$

Let $\mathcal{O}_\kappa$ be the full subcategory of the category of left $\mathfrak{g}$-modules consisting of objects V such that (1) V is locally finite over $\mathfrak{g}_+$, (2) $V = \bigoplus_{\lambda \in \mathfrak{h}_\kappa^*} V^\lambda$ and $\dim_{\mathbb{C}} V^\lambda < \infty$ for all λ, (3) there exists a finite subset $\{ \mu_1, \ldots, \mu_n \} \subset \mathfrak{h}_\kappa^*$ such that $P(V) \subset \bigcup_i \mu_i - Q_+$.

The correspondence $V \rightsquigarrow V^*$ defines the duality functor in $\mathcal{O}_\kappa$. Here, $\mathfrak{g}$ acts on V^* by $(Xf)(v) = f(X^t v)$, where $X \mapsto X^t$ is the Chevalley antiautomorphism of $\mathfrak{g}$.

Let $M(\lambda) \in \mathcal{O}_\kappa$, $\lambda \in \mathfrak{h}_\kappa^*$, be the Verma module of highest weight λ and $L(\lambda)$ its unique simple quotient. Let $\mathcal{O}_\kappa^{[\Lambda]}$, $\Lambda \in \mathfrak{h}_\kappa^*$, be the full subcategory of $\mathcal{O}_\kappa$ whose objects have all their local composition factors isomorphic to $L(w \circ \Lambda)$, $w \in W^\Lambda$. By [20], $\mathcal{O}_\kappa$ splits into the orthogonal direct sum $\mathcal{O}_\kappa = \bigoplus_{\Lambda \in \mathfrak{h}_\kappa^*/\sim} \mathcal{O}_\kappa^{[\Lambda]}$, where $\sim$ is the equivalent relation defined by $\lambda \sim \mu \Leftrightarrow \mu \in W^\lambda \circ \lambda$. Orthogonal here means that $\mathrm{Ext}^i_{\mathcal{O}_\kappa}(M, N) = 0$ for $M \in \mathcal{O}_\kappa^{[\Lambda]}$, $N \in \mathcal{O}_\kappa^{[\Lambda']}$, $i \geq 0$, when $\Lambda \neq \Lambda'$ in $\mathfrak{h}_\kappa^*/\sim$.

2.3. **The Clifford algebra and the Fock space.** Let

$$(7) \qquad\qquad L\bar{\mathfrak{n}}_\pm = \bar{\mathfrak{n}}_\pm \otimes \mathbb{C}[t, t^{-1}] \subset \mathfrak{g}.$$

We identify $L\bar{\mathfrak{n}}_\pm$ with $(L\bar{\mathfrak{n}}_\mp)^*$ using the invariant bilinear form $(\ ,\ )$ of $\mathfrak{g}$. Let $\mathcal{C}l$ be the *Clifford algebra* associated to $L\bar{\mathfrak{n}}_+ \oplus L\bar{\mathfrak{n}}_-$ and its symmetric bilinear form defined by the identification $(L\bar{\mathfrak{n}}_\pm)^* = L\bar{\mathfrak{n}}_\mp$. Denote by $\psi_\alpha(n)$, $\alpha \in \bar{\Delta}$, $n \in \mathbb{Z}$, the generators of $L\bar{\mathfrak{n}}_+ \oplus L\bar{\mathfrak{n}}_- \subset \mathcal{C}l$ which correspond to the elements $J_\alpha(n)$. Then,

$$\{ \psi_\alpha(m), \psi_\beta(n) \} = \delta_{\alpha+\beta,0} \delta_{m+n,0} \quad (\alpha, \beta \in \bar{\Delta}, m, n \in \mathbb{Z}).$$

Here, $\{x, y\} = xy + yx$. The algebra $\mathcal{C}l$ contains the Grassmann algebra $\Lambda(L\bar{\mathfrak{n}}_{\pm})$ of $L\bar{\mathfrak{n}}_{\pm}$ as its subalgebra. We have $\mathcal{C}l = \Lambda(L\bar{\mathfrak{n}}_{+}) \otimes \Lambda(L\bar{\mathfrak{n}}_{-})$ as $\mathbb{C}$-vector spaces. The action of $\widetilde{W}$ on $\mathfrak{g}$ naturally extends to $\mathcal{C}l$: in particular $\bar{P}^{\vee}$ acts as

$$(8) \qquad t_{\mu}(\psi_{\alpha}(n)) = \psi_{\alpha}(n - \langle \alpha, \mu \rangle) \quad (\mu \in \bar{P}^{\vee}, \, \alpha \in \bar{\Delta}, \, n \in \mathbb{Z}).$$

Let $\bar{\mathcal{C}}l$ be the subalgebra of $\mathcal{C}l$ generated by $\psi_{\alpha}(0)$, $\alpha \in \bar{\Delta}$. Then, $\bar{\mathcal{C}}l$ is identified the Clifford algebra associated to the space $\bar{\mathfrak{n}}_{+} \oplus \bar{\mathfrak{n}}_{-}$ and its natural non-degenerate bilinear form. We have:

$$(9) \qquad \bar{\mathcal{C}}l \cong \Lambda(\bar{\mathfrak{n}}_{+}) \otimes \Lambda(\bar{\mathfrak{n}}_{-})$$

as $\mathbb{C}$-vector spaces.

Let $\mathcal{F}(L\bar{\mathfrak{n}}_{\pm})$ be the irreducible representation of $\mathcal{C}l$ generated by a vector $\mathbf{1}$ such that

$$(10) \qquad \psi_{\alpha}(n)\mathbf{1} = 0 \quad (\alpha \in \bar{\Delta}, \, n \in \mathbb{Z}, \, \alpha + n\delta \in \Delta_{+}^{\mathrm{re}}).$$

Thus, $\mathcal{F}(L\bar{\mathfrak{n}}_{\pm}) = \Lambda(L\bar{\mathfrak{n}}_{\mp} \cap \mathfrak{g}_{-}) \otimes \Lambda(L\bar{\mathfrak{n}}_{\pm} \cap \mathfrak{g}_{-})$ as $\mathbb{C}$-vector spaces. Let

$$(11) \qquad \mathcal{F}^{p}(L\bar{\mathfrak{n}}_{\pm}) = \sum_{i-j=p} \Lambda^{i}(L\bar{\mathfrak{n}}_{\mp} \cap \mathfrak{g}_{-}) \otimes \Lambda^{j}(L\bar{\mathfrak{n}}_{\pm} \cap \mathfrak{g}_{-}) \subset \mathcal{F}(L\bar{\mathfrak{n}}_{\pm}) \quad (p \in \mathbb{Z}).$$

Then, $\mathcal{F}(L\bar{\mathfrak{n}}_{\pm}) = \sum_{p \in \mathbb{Z}} \mathcal{F}^{p}(L\bar{\mathfrak{n}}_{\pm})$.

2.4. The BRST complex. For $V \in \mathcal{O}_{\kappa}$, let

$$C(L\bar{\mathfrak{n}}_{\pm}, V) = V \otimes \mathcal{F}(L\bar{\mathfrak{n}}_{\pm}) = \sum_{i \in \mathbb{Z}} C^{i}(L\bar{\mathfrak{n}}_{\pm}, V), \quad \text{where } C^{i}(L\bar{\mathfrak{n}}_{\pm}, V) = V \otimes \mathcal{F}^{i}(L\bar{\mathfrak{n}}_{\pm}).$$

Let $d_{\pm} = d_{\pm}^{\mathrm{st}} + \chi_{\pm} \in \mathrm{End}\, C(L\bar{\mathfrak{n}}_{\pm}, V)$, where

$$(12) \quad d_{\pm}^{\mathrm{st}} = \sum_{\alpha \in \bar{\Delta}_{+}, n \in \mathbb{Z}} J_{\pm\alpha}(-n)\psi_{\mp\alpha}(n) - \frac{1}{2} \sum_{\substack{\alpha, \beta, \gamma \in \bar{\Delta}_{+} \\ k+l+m=0}} c_{\pm\alpha, \pm\beta}^{\pm\gamma} \psi_{\mp\alpha}(k)\psi_{\mp\beta}(l)\psi_{\pm\gamma}(m),$$

$$(13) \quad \chi_{+} = \sum_{\alpha \in \bar{\Pi}} \psi_{-\alpha}(1),$$

$$(14) \quad \chi_{-} = \sum_{\alpha \in \bar{\Pi}} \psi_{\alpha}(0).$$

Then, $(d_{\pm}^{\mathrm{st}})^{2} = \chi_{\pm}^{2} = 0$, $\{d_{\pm}^{\mathrm{st}}, \chi_{\pm}\} = 0$ and $d_{\pm} C^{i}(L\bar{\mathfrak{n}}_{\pm}, V) \subset C^{i+1}(L\bar{\mathfrak{n}}_{\pm}, V)$. In particular, $d_{\pm}^{2} = 0$. For $V \in Obj\mathcal{O}_{\kappa}$, define

$$(15) \qquad H_{\pm}^{\bullet}(V) = H^{\bullet}(C(L\bar{\mathfrak{n}}_{\pm}, V), d_{\pm}).$$

It is called the *cohomology of the BRST complex of the quantized Drinfeld-Sokolov reduction for $L\bar{\mathfrak{n}}_{\pm}$ associated to V* ([8, 10, 11]).

3. $\mathcal{W}$-ALGEBRAS

In this section we collect necessary information about $\mathcal{W}$-algebras. The textbook [10] and the paper [11, 12] are our basic references in this section.

3.1. **The vertex operator algebra $C_\kappa(\bar{\mathfrak{g}})$.** Let

(16) $$V_\kappa(\mathfrak{g}) = U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{g}} \otimes \mathbb{C}[t] \oplus \mathbb{C}K \oplus \mathbb{C}\mathbf{D})} \mathbb{C} \in Obj\mathcal{O}_\kappa.$$

Here, $\mathbb{C}$ is considered as a $\bar{\mathfrak{g}} \otimes \mathbb{C}[t] \oplus \mathbb{C}K \oplus \mathbb{C}\mathbf{D}$-module on which $\bar{\mathfrak{g}} \otimes \mathbb{C}[t] \oplus \mathbb{C}\mathbf{D}$ acts trivially and K acts as $(\kappa - h^\vee)$ id. It is called the *universal affine vertex algebra of level $\kappa - h^\vee$* associated to $\bar{\mathfrak{g}}$. Let

$$C_\kappa(\mathfrak{g}) = C(L\bar{\mathfrak{n}}_+, V_\kappa(\mathfrak{g})) = V_\kappa(\mathfrak{g}) \otimes \mathcal{F}(L\bar{\mathfrak{n}}_+),$$

and let $|0\rangle = (1 \otimes 1) \otimes \mathbf{1}$ be its highest weight vector. The space $C_\kappa(\mathfrak{g})$ has a vertex algebra structure, see [10, Chapter 14]. Let $Y(v,z) \in \operatorname{End} C_\kappa(\mathfrak{g})[[z, z^{-1}]]$ be the field corresponding to $v \in C_\kappa(\mathfrak{g})$. Set

$$X(z) = \sum_{n \in \mathbb{Z}} X(n)z^{-n-1} = Y(X(-1)|0\rangle, z) \quad (X \in \bar{\mathfrak{g}})$$

$$\psi_\alpha(z) = \sum_{n \in \mathbb{Z}} \psi_\alpha(n)z^{-n-1} = Y(\psi_\alpha(-1)|0\rangle, z) \quad (\alpha \in \bar{\Delta}_+),$$

$$\psi_{-\alpha}(z) = \sum_{n \in \mathbb{Z}} \psi_{-\alpha}(n)z^{-n} = Y(\psi_{-\alpha}(0)|0\rangle, z) \quad (\alpha \in \bar{\Delta}_+).$$

We have:

(17) $$[d_+, Y(v,z)] = Y(d_+ v, z) \quad \text{for all } v \in C_\kappa(\mathfrak{g}).$$

Let

$$L(z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2} = L^{\text{tot}}(z) + \partial_z \widehat{h}_{\bar{\rho}^\vee}(z),$$

where

$$L^{\text{tot}}(z) = \sum_{n \in \mathbb{Z}} L^{\text{tot}}(n)z^{-n-2} = L^\mathfrak{g}(z) + L^f(z),$$

$L^\mathfrak{g}(z)$ is the Sugawara filed of $\mathfrak{g}$,

$$L^f(z) = -\sum_{\alpha \in \bar{\Delta}_+} : \psi_\alpha(z)\partial\psi_{-\alpha}(z) :,$$

$$\widehat{h}_{\bar{\rho}^\vee}(z) = \sum_{n \in \mathbb{Z}} \widehat{h}_{\bar{\rho}^\vee}(n)z^{-n-1} = \bar{\rho}^\vee(z) + \sum_{\alpha \in \bar{\Delta}_+} \operatorname{ht} \alpha : \psi_\alpha(z)\psi_{-\alpha}(z) : .$$

Here, $: :$ is the normal ordering in the sence of [10]. Then,

(18) $$[d_+, L(z)] = 0,$$

(19) $$L(z)L(w) \sim \frac{c(\kappa)/2}{(z-w)^4} + \frac{2}{(z-w)^2}L(w) + \frac{1}{z-w}\partial L(w),$$

where

(20) $$c(\kappa) = l - 12\left(\kappa|\bar{\rho}^\vee|^2 - 2\langle\bar{\rho}, \bar{\rho}^\vee\rangle + \frac{1}{\kappa}|\bar{\rho}|^2\right).$$

The OPE (19) is equivalent to the following commutation relations:

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12}\delta_{m+n,0}c(\kappa)\operatorname{id}.$$

Note that

(21) $$L(-1) = L^{\text{tot}}(-1), \quad L(0) = L^{\text{tot}}(0) - \widehat{h}_{\bar{\rho}^\vee}(0), \quad L(1) = L^{\text{tot}}(1) - 2\widehat{h}_{\bar{\rho}^\vee}(1).$$

In particular, $L(0)$ acts on $C_\kappa(\mathfrak{g})$ semisimply and

$$(22) \qquad Y(L(-1)v, z) = \partial_z Y(v, z).$$

Thus, $C_\kappa(\mathfrak{g})$ has the vertex operator algebra structure with Virasoro field $L(z)$.

Let

$$C_\kappa(\mathfrak{g}) = \bigoplus_{\Delta \in \mathbb{Z}} C_\kappa(\mathfrak{g})_\Delta,$$

be the eigenspace decomposition with respect to the action of $L(0)$. Thus,

$$C_\kappa(\mathfrak{g})_\Delta = \bigoplus_{\substack{\lambda \\ \langle \lambda, \mathbf{D} + \bar{\rho}^\vee \rangle = -\Delta}} C_\kappa(\mathfrak{g})^\lambda.$$

Here, $C_\kappa(\mathfrak{g})^\lambda$ is the weight space of weight λ of $C_\kappa(\mathfrak{g})$ with respect to the natural action of $\mathfrak{h}$.

For $v \in C_\kappa(\mathfrak{g})_\Delta$, we expand the corresponding field $Y(v, z)$ as

$$(23) \qquad Y(v, z) = \sum_{n \in \mathbb{Z}} Y_n(v) z^{-n-\Delta}$$

so $[L(0), Y_n(v)] = -nY_n(v)$. Thus, for example, $Y_n(v) = J_\alpha(n - \operatorname{ht} \alpha)$, $\psi_\alpha(n - \operatorname{ht} \alpha)$, $J_{-\alpha}(n + \operatorname{ht} \alpha)$, $\psi_{-\alpha}(n + \operatorname{ht} \alpha)$ for $v = J_\alpha(-1)|0\rangle$, $\psi_\alpha(-1)|0\rangle$, $J_{-\alpha}(-1)|0\rangle$, $\psi_{-\alpha}(0)|0\rangle$ respectively ($n \in \mathbb{Z}$, $\alpha \in \bar{\Delta}_+$).

3.2. **The Zhu algebra of $C_\kappa(\mathfrak{g})$.** For a vertex operator algebra V in general, one associates an associative algebra $\mathcal{A}(V)$ called the *Zhu algebra* ([12]): Let

$$(24) \qquad \mathcal{U}(V) = \bigoplus_{n \in \mathbb{Z}} \mathcal{U}(V)_n, \quad \mathcal{U}(V)_n = \{u \in \mathcal{U}(V); [L(0), u] = -nu\})$$

be the universal enveloping algebra of V in the sense of [12]. According to [12], one can define $\mathcal{A}(V)$ as

$$(25) \qquad \mathcal{A}(V) = \mathcal{U}(V)_0 / \sum_{p>0} \mathcal{U}(V)_{-p} \mathcal{U}(V)_p,$$

(see [21] for a proof of the above identification (25) of $\mathcal{A}(V)$).

The zero-mode algebra $\mathcal{U}_0(C_\kappa(\mathfrak{g})) \subset \mathcal{U}(C_\kappa(\bar{\mathfrak{g}}))$ contains $t_{\bar{\rho}^\vee}(U(\bar{\mathfrak{g}}) \otimes \bar{\mathcal{C}}l) \subset U(\mathfrak{g}) \otimes \mathcal{C}l$ generated by

$$J_{\pm\alpha}(\mp \operatorname{ht} \alpha) = t_{\bar{\rho}^\vee}(J_{\pm\alpha}(0)), \ \psi_{\pm\alpha}(\mp \operatorname{ht} \alpha) = t_{\bar{\rho}^\vee}(\psi_{\pm\alpha}(0)) \ (\alpha \in \bar{\Delta}_+).$$

The following is clear by definition.

Proposition 3.2.1. *For any $\kappa \in \mathbb{C}$, the Zhu algebra $\mathcal{A}(C_\kappa(\mathfrak{g}))$ of $C_\kappa(\mathfrak{g})$ is canonically isomorphic to $t_{\bar{\rho}^\vee}(U(\bar{\mathfrak{g}}) \otimes \bar{\mathcal{C}}l) \cong U(\bar{\mathfrak{g}}) \otimes \bar{\mathcal{C}}l$.*

3.3. **The tensor product decomposition of $C_\kappa(\mathfrak{g})$.** Following [12], we now recall the tensor product decomposition of $C_\kappa(\bar{\mathfrak{g}})$. Set

$$\widehat{J}_a(z) = \sum_{n \in \mathbb{Z}} \widehat{J}_a(n) z^{-n-1} = J_a(z) + \sum_{\beta, \gamma \in \bar{\Delta}_+} c_{a,\beta}^\gamma : \psi_\gamma(z) \psi_{-\beta}(z) :$$

for $a \in \bar{I} \sqcup \bar{\Delta}$. Let $C_\kappa(\mathfrak{g})_0$ be the subspace of $C_\kappa(\mathfrak{g})$ spanned by the elements

$$\widehat{J}_{a_1}(-n_1) \dots \widehat{J}_{a_p}(-n_p) \psi_{-\alpha_{j_1}}(-m_1) \dots \psi_{-\alpha_{j_q}}(-m_q)|0\rangle$$

with $a_i \in \bar{I} \sqcup \bar{\Delta}_-$, $\alpha_{j_i} \in \bar{\Delta}_+$, $n_i, m_i \in \mathbb{Z}$. Similarly, let $C_\kappa(\mathfrak{g})'$ be the subspace of $C_\kappa(\mathfrak{g})$ spanned by the elements

$$\widehat{J}_{\alpha_{i_1}}(-n_1) \ldots \widehat{J}_{\alpha_{i_p}}(-n_p) \psi_{\alpha_{j_1}}(-m_1) \ldots \psi_{\alpha_{j_q}}(-m_q)|0\rangle$$

with $\alpha_{i_s}, \alpha_{j_s} \in \bar{\Delta}_+$, $n_i, m_i \in \mathbb{Z}$. It was shown in [10], generalizing the results of [4], that

(26) $C_\kappa(\mathfrak{g})'$ and $C_\kappa(\mathfrak{g})_0$ are vertex subalgebras of $C_\kappa(\mathfrak{g})$,

(27) $d_+ C_\kappa(\mathfrak{g})' \subset C_\kappa(\mathfrak{g})'$, $d_+ C_\kappa(\mathfrak{g}) \subset C_\kappa(\mathfrak{g})$,

(28) $C_\kappa(\mathfrak{g}) = C_\kappa(\mathfrak{g})' \otimes C_\kappa(\mathfrak{g})_0$ as vertex algebras and complexes,

(29) $H^i(C_\kappa(\mathfrak{g})') = \begin{cases} \mathbb{C} & (i = 0) \\ 0 & (i \neq 0), \end{cases}$

(30) $H^i(C_\kappa(\mathfrak{g})_0) = 0 \; (i \neq 0).$

In particular, by Künneth Theorem, we have

(31) $H^i_+(V_\kappa(\mathfrak{g})) = \begin{cases} H^0(C_\kappa(\mathfrak{g})_0) & (i = 0) \\ 0 & (i \neq 0). \end{cases}$

3.4. **The Zhu algebra of $C_\kappa(\bar{\mathfrak{g}})'$ and $C_\kappa(\bar{\mathfrak{g}})_0$.** The zero-mode algebra $\mathcal{U}_0(C_\kappa(\bar{\mathfrak{g}})')$ of $C_\kappa(\bar{\mathfrak{g}})'$ contains the algebra $t_{\bar{\rho}^\vee}(U(\bar{\mathfrak{n}}_+) \otimes \Lambda(\bar{\mathfrak{n}}_+)) \cong U(\bar{\mathfrak{n}}_+) \otimes \Lambda(\bar{\mathfrak{n}}_+)$ generated by

$$\widehat{J}_\alpha(-\operatorname{ht}\alpha), \psi_\alpha(-\operatorname{ht}\alpha) \quad (\alpha \in \bar{\Delta}_+).$$

Here, the vector space $U(\bar{\mathfrak{n}}_+) \otimes \Lambda(\bar{\mathfrak{n}}_+)$ is considered as an algebra such that (1) the natural maps $U(\bar{\mathfrak{n}}_+) \hookrightarrow U(\bar{\mathfrak{n}}_+) \otimes \Lambda(\bar{\mathfrak{n}}_+)$ and $\Lambda(\bar{\mathfrak{n}}_+) \hookrightarrow U(\bar{\mathfrak{n}}_+) \otimes \Lambda(\bar{\mathfrak{n}}_+)$ are algebra embeddings, and (2) $[u, \omega] = \operatorname{ad} u(\omega)$ $(u \in U(\bar{\mathfrak{n}}_+), \omega \in \Lambda(\bar{\mathfrak{n}}_+))$.

Similarly, $\mathcal{U}_0(C_\kappa(\bar{\mathfrak{g}})_0)$ contains a subalgebra $t_{\bar{\rho}^\vee}(U(\bar{\mathfrak{b}}_-) \otimes \Lambda(\bar{\mathfrak{n}}_-)) \cong U(\bar{\mathfrak{b}}_-) \otimes \Lambda(\bar{\mathfrak{n}}_-)$ generated by

$$\widehat{J}_i(0), \widehat{J}_{-\alpha}(\operatorname{ht}\alpha), \psi_{-\alpha}(\operatorname{ht}\alpha) \quad (i \in \bar{I}, \alpha \in \bar{\Delta}_+).$$

Here, $\bar{\mathfrak{b}}_- = \bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{h}} \subset \bar{\mathfrak{g}}$ and $U(\bar{\mathfrak{b}}_-) \otimes \Lambda(\bar{\mathfrak{n}}_-)$ is considered as an algebra similarly as above.

We have the following proposition.

Proposition 3.4.1. *Let κ be any complex number.*
  (1) $\mathcal{A}(C_\kappa(\bar{\mathfrak{g}})') \cong U(\bar{\mathfrak{n}}_+) \otimes \Lambda(\bar{\mathfrak{n}}_+)$.
  (2) $\mathcal{A}(C_\kappa(\bar{\mathfrak{g}})_0) \cong U(\bar{\mathfrak{b}}_-) \otimes \Lambda(\bar{\mathfrak{n}}_-)$.

3.5. **The tensor product decomposition of $\mathcal{A}(C_\kappa(\mathfrak{g}))$.** Let the differential d_+ act on $\mathfrak{g}(C_\kappa(\mathfrak{g}))$ by $d_+ J(v, f) = J(d_+ v, f)$. This makes $\mathcal{U}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$ and $\mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$ complexes. Similarly, the spaces $\mathcal{U}(C_\kappa(\bar{\mathfrak{g}})')$, $\mathcal{U}(C_\kappa(\bar{\mathfrak{g}})_0)$, $\mathcal{A}(C_\kappa(\bar{\mathfrak{g}})')$ and $\mathcal{A}(C_\kappa(\bar{\mathfrak{g}})_0)$ have the natural structure of complexes.

One can apply the argument of [10, Section 14.2] to prove the following proposition.

Proposition 3.5.1. *Let κ be any complex number.*
  (1) $\mathcal{A}(C_\kappa(\mathfrak{g})) = \mathcal{A}(C_\kappa(\mathfrak{g})') \otimes \mathcal{A}(C_\kappa(\mathfrak{g})_0)$ *as complexes.*
  (2) $H^i(\mathcal{A}(C_\kappa(\mathfrak{g})')) = \begin{cases} \mathbb{C} & (i = 0) \\ 0 & (i \neq 0). \end{cases}$

(3) $H^i(\mathcal{A}(C_\kappa(\mathfrak{g})_0)) = 0$ $(i \neq 0)$.

By Proposition 3.5.1, we conclude as

$$(32) \qquad H^i(\mathcal{A}(C_\kappa(\mathfrak{g}))) = \begin{cases} H^0(\mathcal{A}(C_\kappa(\mathfrak{g})_0)) & (i = 0) \\ 0 & (i \neq 0). \end{cases}$$

3.6. **The $\mathcal{W}$-algebras.** Define

$$(33) \qquad \mathcal{W}_\kappa(\bar{\mathfrak{g}}) = H^0_+(V_\kappa(\mathfrak{g})).$$

By (17), Y descends to a map

$$(34) \qquad Y : \mathcal{W}_\kappa(\bar{\mathfrak{g}}) \to \operatorname{End} \mathcal{W}_\kappa(\bar{\mathfrak{g}})[[z, z^{-1}]].$$

Hence, by (18), $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$ has a vertex operator algebra structure with the Virasoro field $L(z)$ with the central charge $c(\kappa)$. The vertex operator algebra $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$ is called the $\mathcal{W}$-*algebra associated to* $\bar{\mathfrak{g}}$ *at level* $\kappa - h^\vee$. Note that, by (31), we have

$$(35) \qquad \mathcal{W}_\kappa(\bar{\mathfrak{g}}) = H^0(C_\kappa(\mathfrak{g})_0)$$

as vertex algebras.

3.7. **A filtration of $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$.** We now define a decreasing filtration

$$\cdots \supset G^p \mathcal{W}_\kappa(\bar{\mathfrak{g}}) \supset G^{p+1} \mathcal{W}_\kappa(\bar{\mathfrak{g}}) \supset \cdots \supset G^1 \mathcal{W}_\kappa(\bar{\mathfrak{g}}) = \{0\}$$

of $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$ such that

$$(36) \qquad G^p \mathcal{W}_\kappa(\bar{\mathfrak{g}}) \cdot G^q \mathcal{W}_\kappa(\bar{\mathfrak{g}}) \subset G^{p+q} \mathcal{W}_\kappa(\bar{\mathfrak{g}})$$

and the corresponding graded vertex algebra $\operatorname{gr}^G \mathcal{W}_\kappa(\bar{\mathfrak{g}}) = \bigoplus_p G^p \mathcal{W}_\kappa(\bar{\mathfrak{g}})/G^{p+1} \mathcal{W}_\kappa(\bar{\mathfrak{g}})$ is commutative. Here, the left-hand-side of (36) denotes the span of the vectors $Y_k(v_1)v_2$ $(v_1 \in G^p \mathcal{W}_\kappa(\bar{\mathfrak{g}}), v_2 \in G^q \mathcal{W}_\kappa(\bar{\mathfrak{g}}), k \in \mathbb{Z})$.

Set

$$(37) \qquad G^p C^n_\kappa(\mathfrak{g})_0 = \bigoplus_{\substack{\lambda \in \mathfrak{h}^* \\ \langle \lambda, \bar{\rho}^\vee \rangle \geq p - n}} C^n_\kappa(\mathfrak{g})^\lambda_0 \subset C^n_\kappa(\mathfrak{g})_0 \quad (p \leq n + 1),$$

where $C^n_\kappa(\mathfrak{g})_0 = C_\kappa(\mathfrak{g})_0 \cap C^n_\kappa(\mathfrak{g})$. Then,

$$(38) \qquad \cdots \supset G^p C^n_\kappa(\mathfrak{g})_0 \supset G^{p+1} C^n_\kappa(\mathfrak{g})_0 \supset \cdots \supset G^{n+1} C^n_\kappa(\mathfrak{g})_0 = \{0\},$$

$$(39) \qquad C^n_\kappa(\mathfrak{g})_0 = \bigcup_p G^p C^n_\kappa(\mathfrak{g})_0$$

$$(40) \qquad d^{\mathrm{st}}_+ G^p C^n_\kappa(\mathfrak{g})_0 \subset G^{p+1} C^{n+1}_\kappa(\bar{\mathfrak{g}})_0, \quad \chi_+ G^p C^n_\kappa(\bar{\mathfrak{g}})_0 \subset G^p C^{n+1}_\kappa(\bar{\mathfrak{g}})_0.$$

Let $E_r \Rightarrow H^\bullet(C_\kappa(\mathfrak{g})_0, d_+)$ be the corresponding spectral sequence. By definition, $E_1 = H^\bullet(C_\kappa(\mathfrak{g})_0, \chi_+)$.

Let $p_- = \sum_{i \in \bar{I}} \frac{(\alpha_i, \alpha_i)}{2} f_i$. Then, there exists a basis $\{P_i\}_{i \in \bar{I}}$ of $\ker \operatorname{ad} p_{-|\bar{\mathfrak{n}}_-} \subset \bar{\mathfrak{n}}_-$ such that $[\bar{\rho}^\vee, P_i] = -d_i P_i$ $(i \in \bar{I})$, where d_i is the i-th exponent of $\bar{\mathfrak{g}}$. It is known that $\ker \operatorname{ad} p_{-|\bar{\mathfrak{n}}_-} = \operatorname{span}\{P_i\}_{i=1}^l$ is a maximal abelian subalgebra of $\bar{\mathfrak{g}}$.

It was shown in [10, 14.2.8] that

$$(41) \qquad H^i(C_\kappa(\mathfrak{g})_0, \chi_+) = 0 \quad (i \neq 0),$$

$$(42) \qquad \begin{array}{ccc} \mathbb{C}[\widehat{P}_1(-n_1), \ldots, \widehat{P}_l(-n_l)]_{n_1,\ldots,n_l \geq 1} & \cong & H^0(C_\kappa(\mathfrak{g})_0, \chi), \\ f & \mapsto & f|0\rangle \end{array}$$

where $\widehat{P}_i(n)$ is the linear combination of $\widehat{J}_\alpha(n)$ corresponding to P_i. In particular, the spectral sequence collapses at $E_1 = E_\infty$.

Define

(43) $$G^p \mathcal{W}_\kappa(\bar{\mathfrak{g}}) = \mathrm{Im} : H^0(G^p C_\kappa(\mathfrak{g})_0) \hookrightarrow H^0(C_\kappa(\mathfrak{g})_0) = \mathcal{W}_\kappa(\bar{\mathfrak{g}}).$$

Then, by definition,

(44) $$\mathrm{gr}^G \mathcal{W}_\kappa(\bar{\mathfrak{g}}) = E_\infty = H^0(C_\kappa(\mathfrak{g})_0, \chi_+).$$

Since

(45) $$G^p C_\kappa^n(\mathfrak{g})_0 \cdot G^q C_\kappa^m(\mathfrak{g})_0 \subset G^{p+q} C_\kappa^{n+m}(\mathfrak{g})_0,$$

the graded vector space $\mathrm{gr}^G \mathcal{W}_\kappa(\bar{\mathfrak{g}})$ carries a vertex algebra structure Here, the left-hand-side of (45) denotes the span of the vectors $Y_k(v_1)v_2$ ($v_1 \in G^p C^n(\bar{\mathfrak{g}})_0$, $v_2 \in G^q C^m(\bar{\mathfrak{g}})_0$, $k \in \mathbb{Z}$). Since $\mathrm{gr}^G \mathcal{W}_\kappa(\bar{\mathfrak{g}}) \cong H^0(C_\kappa(\bar{\mathfrak{g}})_0, \chi_+)$ is as vertex algebras, $\mathrm{gr}^G \mathcal{W}_\kappa(\bar{\mathfrak{g}})$ is a commutative vertex algebra. Clearly,

(46) $$\mathcal{A}(\mathrm{gr}^G \mathcal{W}_\kappa(\bar{\mathfrak{g}})) \cong \mathcal{A}(H^0(C_\kappa(\mathfrak{g}), \chi_+)) = \mathbb{C}[\widehat{P}_1(d_1), \widehat{P}_2(d_2), \dots, \widehat{P}_l(d_l)].$$

Let W_i be the cocycle in $C_\kappa(\mathfrak{g})_0$ corresponding to $\widehat{P}_i(-1)|0\rangle$. Set

$$W_i(z) = \sum_{n \in \mathbb{Z}} W_i(n) z^{-n-d_i-1} = Y(W_i, z).$$

Then, $W_i(n)|0\rangle = 0$ ($n \geq -d_i$), $W_i = W_i(-d_i-1)|0\rangle$ ($i \in \bar{I}$). By (42), we see that the set

$$\left\{ W_{i_1}(-d_{i_1} - n_1) \dots W_{i_m}(-d_{i_m} - n_m)|0\rangle; \; \begin{array}{l} 1 \leq i_1 \leq \dots \leq i_m \leq l, n_p \geq 1, \\ n_p \geq n_{p+1} \text{ if } i_p = i_{p+1} \end{array} \right\}.$$

forms a $\mathbb{C}$-basis of $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$. Thus, the vertex algebra $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$ is freely generated by the fields $W_1(z), \dots, W_l(z)$ in the sense of [5].

3.8. **A filtration of** $H^0(\mathcal{A}(C_\kappa(\mathfrak{g})_0))$. Similarly as above, we can define a filtration $\{G^p H^0(\mathcal{A}(C_\kappa(\mathfrak{g})_0))\}$ on the $\mathbb{C}$-algebra $H^0(\mathcal{A}(C_\kappa(\mathfrak{g})_0))$. Let $\mathrm{gr}^G H^0(\mathcal{A}(C_\kappa(\mathfrak{g})_0))$ be the corresponding graded algebra. Then, we have

(47) $$\begin{aligned} \mathrm{gr}^G H^0(\mathcal{A}(C_\kappa(\mathfrak{g})_0)) &= H^0(\mathcal{A}(C_\kappa(\mathfrak{g})_0), \chi_+) \\ &= \mathbb{C}[\widehat{P}_1(d_1), \widehat{P}_2(d_2), \dots, \widehat{P}_l(d_l)]. \end{aligned}$$

3.9. **The category** $\mathcal{O}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$. Assume $\kappa \neq 0$. For a $\mathcal{U}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$-module V, let V_h^{gen} be the generalized eigenspace of $L(0)$ of eigenvalue $h \in \mathbb{C}$:

(48) $$V_h^{\mathrm{gen}} = \{v \in V; (L(0) - h)^n v = 0 \text{ for } n \gg 0\}.$$

Let $\mathcal{O}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$ be the full subcategory of $\mathcal{U}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$-modules consisting of objects V such that

(1) there exists a finite set $\{h_1, \dots, h_r\}$ in $\mathbb{C}$ such that $V = \bigoplus_{h \in \bigcup_i (h_i + \mathbb{Z}_{\geq 0})} V_h^{\mathrm{gen}}$,

(2) $\dim V_h^{\mathrm{gen}} < \infty$ for all $h \in \mathbb{C}$.

It is easy to see that the category $\mathcal{O}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$ is abelian.

For an object V in $\mathcal{O}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$, we define the normalized character $\mathrm{ch}\, V$ by

(49) $$\mathrm{ch}\, V = \sum_{h \in \mathbb{C}} q^{h - \frac{c(\kappa)}{24}} \dim_{\mathbb{C}} V_h^{\mathrm{gen}}.$$

3.10. **The functors.** By the definition (33) of $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$, the space $H_+^i(V)$, $V \in Obj\mathcal{O}_\kappa$ is a naturally $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$-module. It is also true that $H_-^i(V)$, $V \in Obj\mathcal{O}_\kappa$, has a $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$-module structure. The action of $\mathcal{U}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$ on $H_-^\bullet(V)$ is twisted by the element $y = w_0 t_{-\bar{\rho}^\vee} = t_{\bar{\rho}^\vee} w_0 \in \widetilde{W}$:

$$(50) \qquad u \cdot v = y(u)v \quad (u \in \mathcal{U}(\mathcal{W}_\kappa(\bar{\mathfrak{g}})), v \in H_-^\bullet(V)).$$

The above action is well-defined since

$$(51) \qquad y(d_+) = d_-,$$

see [11] for the detail.

Assume $\kappa \neq 0$. It was shown in [11] that $H_\pm^\bullet(V)$, $V \in Obj\mathcal{O}_\kappa$, is an object of $\mathcal{O}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$. Thus, we have a family of functors defined by

$$(52) \qquad \begin{array}{ccc} \mathcal{O}_\kappa & \longrightarrow & \mathcal{O}(\mathcal{W}_\kappa(\bar{\mathfrak{g}})) \\ V & \longmapsto & H_\pm^i(V) \end{array} \quad (i \in \mathbb{Z}).$$

4. The Zhu algebra of $\mathcal{W}$-algebras

4.1. **The first isomorphism.** There is an obvious homomorphism $\mathcal{U}(\mathcal{W}_\kappa(\bar{\mathfrak{g}})) \to H^0(\mathcal{U}(C_\kappa(\mathfrak{g})))$, which induces an algebra homomorphism $\mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}})) \to H^0(\mathcal{A}(C_\kappa(\mathfrak{g})))$. On the other hand, by (32) and (35), we have a commutative diagram

$$(53) \qquad \begin{array}{ccc} \mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}})) & =\!=\!= & \mathcal{A}(H^0(C_\kappa(\mathfrak{g})_0))) \\ \downarrow & & \downarrow \\ H^0(\mathcal{A}(C_\kappa(\mathfrak{g}))) & =\!=\!= & H^0(\mathcal{A}(C_\kappa(\mathfrak{g})_0)). \end{array}$$

But the vertical map on the right-hand-side induces a map $\mathcal{A}(\mathrm{gr}^G H^0(C_\kappa(\bar{\mathfrak{g}})_0)) \to \mathrm{gr}^G(H^0(\mathcal{A}(C_\kappa(\bar{\mathfrak{g}}))))$, which is an isomorphism by (46) and (47). Hence the map $\mathcal{A}(H^0(C_\kappa(\mathfrak{g})_0))) \to H^0(\mathcal{A}(C_\kappa(\mathfrak{g})_0))$ is an isomorphism. Therefore we conclude as follows.

Proposition 4.1.1. *For any complex number κ,*

$$\mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}})) \cong H^0(\mathcal{A}(C_\kappa(\mathfrak{g}))) = H^0(\mathcal{A}(C_\kappa(\bar{\mathfrak{g}})_0))$$

as $\mathbb{C}$-algebras.

4.2. **The second isomorphism.** Let

$$\bar{d} = \bar{d}^{\mathrm{st}} + \bar{\chi} \in U(\bar{\mathfrak{g}}) \otimes \bar{\mathcal{C}}l,$$

$$\text{where } \bar{d}^{\mathrm{st}} = \sum_{\alpha \in \bar{\Delta}_+} J_\alpha(0)\psi_{-\alpha}(0) - \frac{1}{2} \sum_{\alpha,\beta,\gamma \in \bar{\Delta}_+} c_{\alpha,\beta}^\gamma \psi_{-\alpha}(0)\psi_{-\beta}(0)\psi_\gamma(0),$$

$$\bar{\chi} = \sum_{\alpha \in \bar{\Pi}} \psi_{-\alpha}(0).$$

Then, $(\bar{d}^{\mathrm{st}})^2 = (\bar{\chi})^2 = 0$, $\{\bar{d}^{\mathrm{st}}, \bar{\chi}\} = 0$. Thus, $(\mathrm{ad}\,\bar{d})^2 = 0$ on $U(\bar{\mathfrak{g}}) \otimes \bar{\mathcal{C}}l$. Here, $U(\bar{\mathfrak{g}}) \otimes \bar{\mathcal{C}}l$ is regarded as a superalgebra and $\mathrm{ad}\,\bar{d}(u)$, $u \in U(\bar{\mathfrak{g}}) \otimes \bar{\mathcal{C}}l$, is the adjoint action on u of the odd element $\bar{d}$. Put $\deg\psi_\alpha(0) = -1$, $\deg\psi_\alpha^*(0) = 1$ ($\alpha \in \bar{\Delta}_+$) and $\deg u = 0$ ($u \in U(\bar{\mathfrak{g}})$). This makes $(U(\bar{\mathfrak{g}}) \otimes \bar{\mathcal{C}}l, \mathrm{ad}\,\bar{d})$ a complex. Note that the corresponding cohomology $H^\bullet(U(\bar{\mathfrak{g}}) \otimes \bar{\mathcal{C}}l, \mathrm{ad}\,\bar{d})$ is naturally a graded $\mathbb{C}$-algebra. The following proposition is straightforward from Proposition 3.2.1.

Proposition 4.2.1. *We have the natural isomorphism of graded $\mathbb{C}$-algebras*

$$H^\bullet(\mathcal{A}(C_\kappa(\mathfrak{g}))) \cong H^\bullet(U(\bar{\mathfrak{g}})\otimes\bar{\mathcal{C}}l, \operatorname{ad}\bar{d}).$$

By (32) and Proposition 4.2.1, it follows that

$$(54) \qquad\qquad H^i(U(\bar{\mathfrak{g}})\otimes\bar{\mathcal{C}}l, \operatorname{ad}\bar{d}) = \{0\} \quad (i \neq 0).$$

Proposition 4.2.2. *The map defined by*

$$\begin{array}{ccc}
\mathcal{Z}(\bar{\mathfrak{g}}) & \rightarrow & H^0(U(\bar{\mathfrak{g}})\otimes\bar{\mathcal{C}}l, \operatorname{ad}\bar{d}) \\
z & \mapsto & z\otimes 1
\end{array}$$

is an isomorphism of $\mathbb{C}$-algebras.

Proof. **Step 1)** In what follows we shall identify $\Lambda(\bar{\mathfrak{n}}_-)$ with $\Lambda(\bar{\mathfrak{n}}_+^*)$. Thus, $\bar{\mathcal{C}}l = \Lambda(\bar{\mathfrak{n}}_+^*)\otimes\Lambda(\bar{\mathfrak{n}}_+)$ as a $\mathbb{C}$-vector space. Set $\psi_\alpha = \psi_\alpha(0) \in \Lambda(\bar{\mathfrak{n}}_+)$ and $\psi_\alpha^* = \psi_{-\alpha}(0) \in \Lambda(\bar{\mathfrak{n}}_+^*)$. Put

$$\bar{C} = \sum_{n\in\mathbb{Z}} \bar{C}^n = U(\bar{\mathfrak{g}})\otimes\bar{\mathcal{C}}l.$$

Here,

$$(55) \qquad\qquad \bar{C}^n = \sum_{i-j=n} U(\bar{\mathfrak{g}})\otimes\Lambda^i(\bar{\mathfrak{n}}_+^*)\otimes\Lambda^j(\bar{\mathfrak{n}}_+).$$

We have

$$(56) \qquad\qquad \operatorname{ad}\bar{d} = \bar{d}_+ + \bar{d}_-$$

on $\bar{C}$, where $\bar{d}_\pm$ is defined by

$$\bar{d}_+(u\otimes\omega_1\otimes\omega_2) = \sum_{\alpha\in\bar{\Delta}_+} \{((\operatorname{ad}J_\alpha(u))\otimes\psi_\alpha^*\omega_1\otimes\omega_2 + u\otimes\psi_\alpha^*\omega_1\otimes\operatorname{ad}J_\alpha(\omega_2)\}$$

$$(57)$$

$$- \frac{1}{2}\sum_{\alpha,\beta,\gamma\in\bar{\Delta}_+} c_{\alpha,\beta}^\gamma u\otimes\psi_\alpha^*\psi_\beta^*(\operatorname{ad}\psi_\gamma(\omega_1))\otimes\omega_2,$$

$$(-1)^i\bar{d}_-(u\otimes\omega_1\otimes\omega_2) = \sum_{\alpha\in\bar{\Delta}_+} u(J_\alpha + \bar{\chi}(J_\alpha))\otimes\omega_1\otimes\operatorname{ad}\psi_\alpha^*(\omega_2)$$

$$(58)$$

$$- \frac{1}{2}\sum_{\alpha,\beta,\gamma\in\bar{\Delta}_+} c_{\alpha,\beta}^\gamma u\otimes\omega_1\otimes\psi_\gamma\operatorname{ad}\psi_\beta^*(\operatorname{ad}\psi_\alpha^*(\omega_2))$$

for $u \in U(\bar{\mathfrak{g}})$, $\omega_1 \in \Lambda^i(\bar{\mathfrak{n}}_+^*)$, $\omega_2 \in \Lambda(\bar{\mathfrak{n}}_+)$. Note that

$$(59) \qquad \begin{aligned} &\bar{d}_+(U(\bar{\mathfrak{g}})\otimes\Lambda^i(\bar{\mathfrak{n}}_+^*)\otimes\Lambda^j(\bar{\mathfrak{n}}_+)) \subset U(\bar{\mathfrak{g}})\otimes\Lambda^{i+1}(\bar{\mathfrak{n}}_+^*)\otimes\Lambda^j(\bar{\mathfrak{n}}_+), \\ &\bar{d}_-(U(\bar{\mathfrak{g}})\otimes\Lambda^i(\bar{\mathfrak{n}}_+^*)\otimes\Lambda^j(\bar{\mathfrak{n}}_+)) \subset U(\bar{\mathfrak{g}})\otimes\Lambda^i(\bar{\mathfrak{n}}_+^*)\otimes\Lambda^{j-1}(\bar{\mathfrak{n}}_+). \end{aligned}$$

Hence, it follows that

$$(60) \qquad\qquad \bar{d}_+^2 = \bar{d}_-^2 = \{\bar{d}_+, \bar{d}_-\} = 0.$$

Set

$$(61) \qquad\qquad F^p\bar{C} = \sum_{i\geq p} U(\bar{\mathfrak{g}})\otimes\Lambda^i(\bar{\mathfrak{n}}_+^*)\otimes\Lambda^\bullet(\bar{\mathfrak{n}}_+) \subset \bar{C},$$

Then,

$$\bar{C} = F^0\bar{C} \supset F^1\bar{C} \supset \cdots \supset F^{\dim\bar{\mathfrak{n}}_+ +1}\bar{C} = \{0\}, \tag{62}$$

$$\bigcap F^p\bar{C} = \{0\}, \tag{63}$$

$$d_+ F^p\bar{C} \subset F^{p+1}\bar{C}, \quad d_- F^p\bar{C} \subset F^p\bar{C}. \tag{64}$$

We consider the spectral sequence E such that

$$E_0^{p,q} = F^p\bar{C}^{p+q}/F^{p+1}\bar{C}^{p+q}.$$

Step 2) We have $E_1 = H^\bullet(\bar{C}, \bar{d}_-)$. Let $\mathbb{C}_{\bar{\chi}}$ be the one-dimensional representation of $U(\bar{\mathfrak{n}}_+)$ defined by the character $\bar{\chi}$. Then, by (58), we see that the complex $(\bar{C}, \bar{d}_-)$ is nothing but the Chevalley complex for calculating the $\bar{\mathfrak{n}}_+$-homology $H_\bullet(\bar{\mathfrak{n}}_+, (U(\bar{\mathfrak{g}})\otimes\mathbb{C}_{\bar{\chi}})\otimes\Lambda(\bar{\mathfrak{n}}_+^*))$ (with the opposite grading). Here, $(U(\bar{\mathfrak{g}})\otimes\mathbb{C}_{\bar{\chi}})\otimes\Lambda(\bar{\mathfrak{n}}_+^*)$ is regarded as a right $U(\bar{\mathfrak{n}}_+)$-module on which $U(\bar{\mathfrak{n}}_+)$ acts on the first factor $U(\bar{\mathfrak{g}})\otimes\mathbb{C}_{\bar{\chi}}$. But obviously, $U(\bar{\mathfrak{g}})$ is free over $\bar{\mathfrak{n}}_+$, thus so is $U(\bar{\mathfrak{g}})\otimes\mathbb{C}_{\bar{\chi}}$. Therefore,

$$\begin{aligned}
E_1^{\bullet,q} &= \begin{cases} \left[(U(\bar{\mathfrak{g}})\otimes\mathbb{C}_{\bar{\chi}})\,/\,(U(\bar{\mathfrak{g}})\otimes\mathbb{C}_{\bar{\chi}})\,\bar{\mathfrak{n}}_+\right]\otimes\Lambda(\bar{\mathfrak{n}}_+^*) & (q=0) \\ 0 & (q\neq 0) \end{cases} \\
&= \begin{cases} \left(U(\bar{\mathfrak{g}})\otimes_{U(\bar{\mathfrak{n}}_+)}\mathbb{C}_{-\bar{\chi}}\right)\otimes\Lambda(\bar{\mathfrak{n}}_+^*) & (q=0) \\ 0 & (q\neq 0). \end{cases}
\end{aligned} \tag{65}$$

Note that the space $U(\bar{\mathfrak{g}})\otimes_{U(\bar{\mathfrak{n}}_+)}\mathbb{C}_{-\bar{\chi}}$ has a natural left $\bar{\mathfrak{n}}_+$-module structure defined by $x\cdot u\otimes 1 = (xu)\otimes 1$, $x\in\bar{\mathfrak{n}}_+, u\in U(\bar{\mathfrak{g}})$.

Step 3) Next we calculate $E_2 = H^\bullet(H^\bullet(\bar{C}, \bar{d}_-), \bar{d}_+)$. The formula (57) and (65) show that

$$E_2^{p,0} = H^p(\bar{\mathfrak{n}}_+, \left(U(\bar{\mathfrak{g}})\otimes_{U(\bar{\mathfrak{n}}_+)}\mathbb{C}_{-\bar{\chi}}\right)\otimes\mathbb{C}_{\bar{\chi}}).$$

Here, the action of $\bar{\mathfrak{n}}_+$ on $\left(U(\bar{\mathfrak{g}})\otimes_{U(\bar{\mathfrak{n}}_+)}\mathbb{C}_{-\bar{\chi}}\right)\otimes\mathbb{C}_{\bar{\chi}}$ is the tensor product action. But it is well-known since Kostant [19] that $H^i(\bar{\mathfrak{n}}_+, \left(U(\bar{\mathfrak{g}})\otimes_{U(\bar{\mathfrak{n}}_+)}\mathbb{C}_{-\bar{\chi}}\right)\otimes\mathbb{C}_{\bar{\chi}}) = 0$ $(i\neq 0)$ and we have an isomorphism

$$\begin{aligned}
\mathcal{Z}(\bar{\mathfrak{g}}) \quad&\cong\quad E_2^{0,0} = H^0(\bar{\mathfrak{n}}_+, \left(U(\bar{\mathfrak{g}})\otimes_{U(\bar{\mathfrak{n}}_+)}\mathbb{C}_{-\bar{\chi}}\right)\otimes\mathbb{C}_{\bar{\chi}}), \\
z \quad&\mapsto\quad (z\otimes 1)\otimes 1.
\end{aligned}$$

Hence the spectral sequence collapse at $E_2 = E_\infty$ and Proposition is proved. $\qquad\square$

By Proposition 4.1.1 and Proposition 4.2.2, we conclude as the following.

Theorem 4.2.3. *For any complex number κ, the Zhu algebra $\mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$ of $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$ is naturally isomorphic to $\mathcal{Z}(\bar{\mathfrak{g}})$.*

Remark 4.2.4. Let $\kappa\neq 0$. Then, one calculates that, under the isomorphism $\mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))\cong\mathcal{Z}(\bar{\mathfrak{g}})$, the image of $L(0)$ in $\mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$ is mapped to the element

$$\frac{1}{2\kappa}\Omega - \frac{1}{2}\left(\kappa|\bar{\rho}^\vee|^2 - 2\langle\bar{\rho}, \bar{\rho}^\vee\rangle\right)\mathrm{id}$$

of $\mathcal{Z}(\bar{\mathfrak{g}})$. Here, Ω is the Casimir element of $U(\bar{\mathfrak{g}})$.

5. Simple Modules over $\mathcal{W}$-algebras

5.1. **Verma modules of $\mathcal{W}$-algebra.** In what follows we identify $\mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$ with $\mathcal{Z}(\bar{\mathfrak{g}})$ by Theorem 4.2.3. Let $\gamma : \mathcal{Z}(\bar{\mathfrak{g}}) \cong S(\bar{\mathfrak{h}})^{\bar{W}}$ be the Harish-Chandra isomorphism. For $\bar{\lambda} \in \bar{\mathfrak{h}}^*$, let

$$\gamma_{\bar{\lambda}} = (\text{evaluation at } \bar{\lambda} + \bar{\rho}) \circ \gamma : \mathcal{Z}(\bar{\mathfrak{g}}) \to \mathbb{C}.$$

Thus, $z \in \mathcal{Z}(\bar{\mathfrak{g}})$ acts as $\gamma_{\bar{\lambda}}(z)\,\mathrm{id}$ on the Verma module of $\bar{\mathfrak{g}}$ of highest weight $\bar{\lambda}$. Let $\mathbb{C}_{\gamma_{\bar{\lambda}}}$ be one-dimensional representation of $\mathcal{Z}(\bar{\mathfrak{g}}) = \mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$ defined by $\gamma_{\bar{\lambda}}$. We also regard $\mathbb{C}_{\gamma_{\bar{\lambda}}}$ as a $\mathcal{U}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))_{\geq 0}$-module on which $\mathcal{U}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))_n$, $n > 0$, acts trivially. Define

(66) $$\mathbf{M}(\gamma_{\bar{\lambda}}) = \mathcal{U}(\mathcal{W}_\kappa(\bar{\mathfrak{g}})) \otimes_{\mathcal{U}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))_{\geq 0}} \mathbb{C}_{\gamma_{\bar{\lambda}}}.$$

The module $\mathbf{M}(\gamma_{\bar{\lambda}})$ is called the *Verma module of $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$ of highest weight $\gamma_{\bar{\lambda}}$*. Let $|\gamma_{\bar{\lambda}}\rangle$ denote the vector $1 \otimes 1 \in \mathbf{M}(\gamma_{\bar{\lambda}})$. By (20) and Remark 4.2.4, we have:

(67) $$L(0)|\gamma_{\bar{\lambda}}\rangle = \Delta_{\bar{\lambda}}|\gamma_{\bar{\lambda}}\rangle \quad (\text{if } \kappa \neq 0),$$

where

(68) $$\Delta_{\bar{\lambda}} = \frac{|\bar{\lambda} + \bar{\rho}|^2}{2\kappa} - \frac{\mathrm{rank}\,\bar{\mathfrak{g}}}{24} + \frac{c(\kappa)}{24}.$$

Let

$$G^p \mathbf{M}(\gamma_{\bar{\lambda}}) = \sum_{p_1 + \cdots + p_r \geq p} G^{p_1}\mathcal{W}_\kappa(\bar{\mathfrak{g}}) \ldots G^{p_r}\mathcal{W}_\kappa(\bar{\mathfrak{g}})|\gamma_{\bar{\lambda}}\rangle \subset \mathbf{M}(\gamma_{\bar{\lambda}}) \quad (p \leq 1).$$

Then,

(69) $$\cdots \supset G^p\mathbf{M}(\gamma_{\bar{\lambda}}) \supset G^{p+1}\mathbf{M}(\gamma_{\bar{\lambda}}) \supset \cdots \supset G^0\mathbf{M}(\gamma_{\bar{\lambda}}) \supset G^1\mathbf{M}(\gamma_{\bar{\lambda}}) = \{0\},$$

(70) $$\mathbf{M}(\gamma_{\bar{\lambda}}) = \bigcup_p G^p\mathbf{M}(\gamma(\bar{\lambda}))$$

(71) $$G^p\mathcal{W}_\kappa(\bar{\mathfrak{g}}) \cdot G^q\mathbf{M}(\gamma_{\bar{\lambda}}) \subset G^{p+q}\mathbf{M}(\gamma_{\bar{\lambda}}).$$

Here, in (71), $G^p\mathcal{W}_\kappa(\bar{\mathfrak{g}}) \cdot G^q\mathbf{M}(\gamma_{\bar{\lambda}})$ denotes the span of the vectors $Y_n(v)m$ ($n \in \mathbb{Z}$, $v \in G^p\mathcal{W}_\kappa(\bar{\mathfrak{g}})$ $m \in G^q\mathbf{M}(\gamma_{\bar{\lambda}})$). Let $\mathrm{gr}^G \mathbf{M}(\gamma_{\bar{\lambda}})$ be the corresponding graded vector space. By (71), $\mathrm{gr}^G \mathbf{M}(\gamma_{\bar{\lambda}})$ is a module over the commutative vertex algebra $\mathrm{gr}^G \mathcal{W}_\kappa(\bar{\mathfrak{g}})$. Let $\bar{W}_i(n)$ denote the image of $W_i(n)$ in $\mathrm{gr}^G \mathcal{W}_\kappa(\bar{\mathfrak{g}})$. The following proposition is easy to see.

Proposition 5.1.1. *For all $\bar{\lambda} \in \bar{\mathfrak{h}}^*$, we have the isomorphism*

$$\begin{array}{ccc} \mathbb{C}[\bar{W}_1(-n_1), \ldots, \bar{W}_l(-n_l)]_{n_1,\ldots,n_l \geq 1} & \cong & \mathrm{gr}^G \mathbf{M}(\gamma_{\bar{\lambda}}) \\ f & \mapsto & f\overline{|\gamma_{\bar{\lambda}}\rangle} \end{array}$$

Here, $\overline{|\gamma_{\bar{\lambda}}\rangle}$ is the image of $|\gamma_{\bar{\lambda}}\rangle$ in $\mathrm{gr}^G \mathbf{M}(\gamma_{\bar{\lambda}})$.

Corollary 5.1.2. *For $\bar{\lambda} \in \bar{\mathfrak{h}}^*$, the set*

$$\left\{ W_{i_1}(-n_1) \ldots W_{i_m}(-n_m)|\gamma_{\bar{\lambda}}\rangle; \begin{array}{l} 1 \leq i_1 \leq \cdots \leq i_m \leq l, n_p \geq 1, \\ n_p \geq n_{p+1} \text{ if } i_p = i_{p+1} \end{array} \right\}.$$

forms a basis of $\mathbf{M}(\gamma_{\bar{\lambda}})$. In particular, for $\kappa \in \mathbb{C}^*$, $\mathbf{M}(\gamma_{\bar{\lambda}}) \in \mathcal{O}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$ and

$$\mathrm{ch}\,\mathbf{M}(\gamma_{\bar{\lambda}}) = \frac{q^{\frac{|\bar{\lambda} + \bar{\rho}|^2}{2\kappa}}}{\eta(\tau)^{\mathrm{rank}\,\bar{\mathfrak{g}}}}.$$

where $\eta(\tau) = q^{\frac{1}{24}} \prod_{i \geq 1}(1 - q^i)$, $q = e^{2\pi\sqrt{-1}\tau}$.

Assume $\kappa \neq 0$. By Corollary 5.1.2, we have

$$(72) \qquad \mathbf{M}(\gamma_{\bar{\lambda}}) = \bigoplus_{\Delta \in \Delta_{\bar{\lambda}} + \mathbb{Z}_{\geq 0}} \mathbf{M}(\gamma_{\bar{\lambda}})_\Delta, \quad \mathbf{M}(\gamma_{\bar{\lambda}})_{\Delta_{\bar{\lambda}}} = \mathbb{C}|\gamma_{\bar{\lambda}}\rangle.$$

Therefore, $N_{\Delta_{\bar{\lambda}}} = \{0\}$ for any proper submodule N of $\mathbf{M}(\gamma_{\bar{\lambda}})$. Hence, the Verma module $\mathbf{M}(\gamma_{\bar{\lambda}})$ has a unique simple quotient which we shall denote by $\mathbf{L}(\gamma_{\bar{\lambda}})$. The module $\mathbf{L}(\gamma_{\bar{\lambda}})$ is called the *irreducible* $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$-*module of highest weight* $\gamma_{\bar{\lambda}}$. The following theorem is clear by [12, Theorem 1.4.2].

Theorem 5.1.3. *For any* $\kappa \in \mathbb{C}^*$, *the set*

$$\{\mathbf{L}(\gamma_{\bar{\lambda}}); \bar{\lambda} \in \bar{\mathfrak{h}}^* / \sim\}$$

is a complete set of isomorphism classes of simple modules in $\mathcal{O}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$. *Here,* $\sim$ *is an equivalence relation defined by* $\bar{\lambda} \sim \bar{\mu} \iff \bar{\mu} + \bar{\rho} \in \bar{W}(\bar{\lambda} + \bar{\rho})$ $(\bar{\lambda}, \bar{\mu} \in \bar{\mathfrak{h}}^*)$.

5.2. **The duality functor** D. Assume that $\kappa \neq 0$ so that $L(z)$ is well-defined.

For $V \in \mathcal{O}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$ (or for simply a $L(0)$-module V), let

$$(73) \qquad D(V) = \bigoplus_{h \in \mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(V_h^{\mathrm{gen}}, \mathbb{C}).$$

Let

$$(74) \qquad \theta(Y_n(v)) = (-1)^\Delta \sum_{j \geq 0} \frac{1}{j!} Y_{-n}(L(1)^j v) \quad \text{for } v \in \mathcal{W}_\kappa(\bar{\mathfrak{g}})_\Delta \text{ and } n \in \mathbb{Z}.$$

Then, as is known, the space $D(V)$ has a $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$-module structure defined by $\langle Y_n(v)f, v_1\rangle = \langle f, \theta(Y_n(v))v_1\rangle$ $(v \in \mathcal{W}_\kappa(\bar{\mathfrak{g}}), f \in D(V), v_1 \in V)$. The correspondence $V \rightsquigarrow D(V)$ defines a duality functor in $\mathcal{O}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$.

It is known that the map θ induces an anti-automorphism of $\mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$, which is also denoted by θ.

Lemma 5.2.1. *Let* $\bar{\theta}$ *be the anti-automorphism of* $U(\bar{\mathfrak{g}})$ *defined by*

$$\bar{\theta}(J_{\pm\alpha}) = -(-1)^{\mathrm{ht}\,\alpha} J_{\pm\alpha} \ (\alpha \in \bar{\Delta}_+), \quad \bar{\theta}(h) = -h \ (h \in \bar{\mathfrak{h}}).$$

Then, $\theta(z) = \bar{\theta}(z)$ *for* $z \in \mathcal{Z}(\bar{\mathfrak{g}})$ *under the identification* $\mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}})) = \mathcal{Z}(\bar{\mathfrak{g}})$.

Proof. Straightforward from the definition. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 5.2.2. *For* $\bar{\lambda} \in \bar{\mathfrak{h}}^*$, $\gamma_{\bar{\lambda}} \circ \bar{\theta}_{|\mathcal{Z}(\bar{\mathfrak{g}})} = \gamma_{-w_0(\bar{\lambda})}$.

Proof. Since $-w_0(\lambda) + \bar{\rho} = -w_0(\lambda + \bar{\rho})$, it is sufficient to show that

$$(75) \qquad\qquad (\gamma \circ \bar{\theta}(z))(\lambda) = \gamma(z)(-\lambda) \quad (\text{for } z \in \mathcal{Z}(\bar{\mathfrak{g}}))$$

for all $\lambda \in \bar{\mathfrak{h}}^*$. But certainly (75) holds for all $\lambda \in \bar{P}_+ + \bar{\rho}$, where $\bar{P}_+ = \{\lambda \in \bar{P}; \langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \bar{\Delta}_+\}\}$. Therefore, (75) holds for all $\lambda \in \bar{\mathfrak{h}}^*$. $\qquad\square$

Lemma 5.2.1 and Lemma 5.2.2 imply the following result.

Theorem 5.2.3. *For all* $\bar{\lambda} \in \bar{\mathfrak{h}}^*$, *we have* $D(\mathbf{L}(\gamma_{\bar{\lambda}})) \cong \mathbf{L}(\gamma_{-w_0(\bar{\lambda})})$.

The functor D is clearly exact. Thus, by Theorem 5.2.3, the exact sequence $\mathbf{M}(\gamma_{-w_0(\bar\lambda)}) \to \mathbf{L}(\gamma_{-w_0(\bar\lambda)}) \to 0$ gives rise to the exact sequence

$$0 \to \mathbf{L}(\gamma_{\bar\lambda}) \to D(\mathbf{M}(\gamma_{-w_0(\bar\lambda)})). \tag{76}$$

Thus, $D(\mathbf{M}(\gamma_{-w_0(\bar\lambda)}))$ has $\mathbf{L}(\gamma_{\bar\lambda})$ as its unique simple submodule. The following lemma will be used in next sections.

Lemma 5.2.4. *Let V be an object of $\mathcal{O}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$. Suppose there exist exact sequences*

$$\mathbf{M}(\gamma_{\bar\lambda}) \to V \to 0, \tag{77}$$
$$0 \to V \to D(\mathbf{M}(\gamma_{-w_0(\bar\lambda)})). \tag{78}$$

Then, V is either $\{0\}$ or isomorphic to $\mathbf{L}(\gamma_{\bar\lambda})$.

Proof. Let N be a proper submodule of V. Then, (77) implies that $N_{\Delta_{\bar\lambda}} = \{0\}$. But then, by (78), it follows that $N = \{0\}$. Lemma is proved. $\square$

Now, for $\bar\lambda \in \bar{\mathfrak{h}}^*$, set

$$G^p D(\mathbf{M}(\gamma_{\bar\lambda})) = D(\mathbf{M}(\gamma_{\bar\lambda})/G^{-p}\mathbf{M}(\gamma_{\bar\lambda})) \quad \text{(as a vector space).}$$

Then,

$$D(\mathbf{M}(\gamma_{\bar\lambda})) = G^{-1}D(\mathbf{M}(\gamma_{\bar\lambda})) \supset G^0 D(\mathbf{M}(\gamma_{\bar\lambda})) \supset \cdots \supset G^p D(\mathbf{M}(\gamma_{\bar\lambda})) \supset \ldots,$$
$$\bigcap_p G^p D(\mathbf{M}(\gamma_{\bar\lambda})) = \{0\},$$
$$G^p \mathcal{W}_\kappa(\bar{\mathfrak{g}}) \cdot G^q D(\mathbf{M}(\gamma_{\bar\lambda})) \subset G^{p+q} D(\mathbf{M}(\gamma_{\bar\lambda})).$$

Let $\mathrm{gr}^G D(\mathbf{M}(\gamma_{\bar\lambda}))$ be the corresponding graded vector space. Then,

$$\mathrm{gr}^G D(\mathbf{M}(\gamma_{\bar\lambda})) = \left(\mathrm{gr}^G \mathbf{M}(\gamma_{\bar\lambda})\right)^*,$$

where * denotes the graded dual in the obvious sense. The space $\mathrm{gr}^G D(\mathbf{M}(\gamma_{\bar\lambda}))$ is a module over $\mathrm{gr}^G \mathcal{W}_\kappa(\bar{\mathfrak{g}})$, on which $\overline{W}_i(n)$ ($i \in \bar{I}$, $n \in \mathbb{Z}$) acts as

$$(\overline{W}_i(n)f)(v) = -(-1)^{d_i} f(\overline{W}_i(-n)v) \quad (f \in \mathrm{gr}^G D(\mathbf{M}(\gamma_{\bar\lambda})), v \in \mathrm{gr}^G \mathbf{M}(\gamma_{\bar\lambda})). \tag{79}$$

In particular, it follows that

$$\{v \in \mathrm{gr}^G D(\mathbf{M}(\gamma_{\bar\lambda})); \bar{W}_i(n)v = 0 \quad (i \in \bar{I}, n > 0)\} = \mathbb{C}\overline{|\gamma_{\bar\lambda}\rangle}^*. \tag{80}$$

Here, $\overline{|\gamma_{\bar\lambda}\rangle}^*$ is the image of the vector $|\gamma_{\bar\lambda}\rangle^*$ dual to $|\gamma_{\bar\lambda}\rangle$.

6. Quantized reductions and the representation theory of $\mathcal{W}$-algebras

In this section we assume that $\kappa \in \mathbb{C}^*$ unless otherwise stated.

6.1. **The images of Verma modules.** For an object V of $\mathcal{O}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$, let

$$
(81) \qquad V_{\text{top}} = \sum_{\substack{h \in \mathbb{C} \\ V_{h-n}^{\text{gen}} = \{0\} \text{ for all } n > 0}} V_h^{\text{gen}} \subset V.
$$

Then, $\mathcal{U}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))_n \cdot V_{\text{top}} = \{0\}$ for $n > 0$. Thus, V_{top} is naturally an $\mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$-module.

By [1, Theorem 5.7, Remark 5.8], we have:

$$
(82) \qquad H_\pm^i(M(\lambda)) = \{0\} \quad (i \neq 0),
$$

$$
(83) \qquad \operatorname{ch} H_-^0(M(\lambda)) = \operatorname{ch} \mathbf{M}(\gamma_{\bar{\lambda}}),
$$

$$
(84) \qquad \operatorname{ch} H_+^0(M(\lambda)) = \operatorname{ch} \mathbf{M}(\gamma_{\overline{t_{-\bar{\rho}^\vee} \circ \lambda}}),
$$

for all $\lambda \in \mathfrak{h}^*$. By (84), one sees that

$$
(85) \qquad H_-^0(M(\lambda))_{\text{top}} = H_-^0(M(\lambda))_{\Delta_{\bar{\lambda}}} = \mathbb{C}|\lambda\rangle,
$$

$$
(86) \qquad H_+^0(M(\lambda))_{\text{top}} = H_+^0(M(\lambda))_{\Delta_{\overline{t_{-\bar{\rho}^\vee} \circ \lambda}}} = \mathbb{C}|\lambda\rangle.
$$

Here, $|\lambda\rangle = v_\lambda \otimes \mathbf{1}$ and v_λ is the highest weight vector of $M(\lambda)$.

Lemma 6.1.1.

    (1) *For any $\lambda \in \mathfrak{h}^*$, $H_-^0(M(\lambda))_{\text{top}} \cong \mathbb{C}_{\gamma_{\bar{\lambda}}}$ as $\mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$-modules.*
    (2) *For any $\lambda \in \mathfrak{h}^*$, $H_+^0(M(\lambda))_{\text{top}} \cong \mathbb{C}_{\gamma_{\overline{t_{-\bar{\rho}^\vee} \circ \lambda}}}$ as $\mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$-modules.*

Proof. (1) It is not difficult to see that the center $\mathcal{Z}(\bar{\mathfrak{g}})$ naturally acts on the space $H_-^0(M(\lambda))_{\text{top}} = \mathbb{C}|\lambda\rangle$ and that its action coincides with the action of $\mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$ under the identification $\mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}})) = \mathcal{Z}(\bar{\mathfrak{g}})$. (2) Let $|\lambda\rangle_\pm$ temporary denote the vector $v_\lambda \otimes \mathbf{1} \in M(\lambda) \otimes \mathcal{F}(L\bar{\mathfrak{n}}_\pm)$. Since $\mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}})) = H^0(\mathcal{A}(C_\kappa(\bar{\mathfrak{g}})_0))$ by Proposition 4.1.1, it is sufficient to show that $\mathbb{C}|\lambda\rangle_+ \cong \mathbb{C}|y \circ \lambda\rangle_-$ as $\mathcal{A}(C_\kappa(\bar{\mathfrak{g}})_0)$-modules. Here, y is defined in Section 3.10. But $\mathcal{A}(C_\kappa(\bar{\mathfrak{g}})_0)$ is generated by $\widehat{J}_i(0)$ $(i \in \bar{I})$, $\widehat{J}_{-\alpha}(\operatorname{ht}\alpha)$, $\psi_{-\alpha}(\operatorname{ht}\alpha)$ $(\alpha \in \bar{\Delta}_+)$, and both spaces $\mathbb{C}|\lambda\rangle_+$ and $\mathbb{C}|\lambda\rangle_-$ are $\mathcal{A}(C_\kappa(\bar{\mathfrak{g}})_0)$-modules on which $\widehat{J}_{-\alpha}(\operatorname{ht}\alpha)$, $\psi_{-\alpha}(\operatorname{ht}\alpha)$ $(\alpha \in \bar{\Delta}_+)$ act trivially. Hence, $\mathbb{C}|\lambda\rangle_+ \cong \mathbb{C}|y \circ \lambda\rangle_-$ as $\mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}})_0)$-modules by [1, Proposition 4.2]. $\qquad\square$

Now we have the following result.

Theorem 6.1.2. *For any $\lambda \in \mathfrak{h}^*$,*

$$
H_-^0(M(\lambda)) \cong \mathbf{M}(\gamma_{\bar{\lambda}}).
$$

Proof. By Lemma 6.1.1 (1), we have a homomorphism $\phi : \mathbf{M}(\gamma_{\bar{\lambda}}) \to \mathcal{F}(M(\lambda))$ of $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$-modules define by $\phi(|\gamma_{\bar{\lambda}}\rangle) = |\lambda\rangle$. Hence, it is sufficient to show that ϕ is a bijection. This follows from the following claim:

Claim There exists a decreasing filtration $\{G^p H_-^0(M(\lambda))\}_{p \leq 1}$ of $H_-^0(M(\lambda))$ such that

$$
\cdots \supset G^p H_-^0(M(\lambda)) \supset G^{p+1} H_-^0(M(\lambda)) \supset \cdots \supset G^1 H_-^0(M(\lambda)) = \{0\},
$$

$$
H_-^0(M(\lambda)) = \bigcup_{p \leq 0} G^p H_-^0(M(\lambda)),
$$

$$
\phi(G^p(\mathbf{M}(\gamma_{\bar{\lambda}}))) \subset G^p H_-^0(M(\lambda)) \quad (\forall p),
$$

and the induced map $\bar{\phi} : \mathrm{gr}^G \mathbf{M}(\gamma_{\bar{\lambda}}) \to \mathrm{gr}^G H^0_-(M(\lambda))$ is an isomorphism of $\mathrm{gr}^G \mathcal{W}_\kappa(\bar{\mathfrak{g}})$-modules.

The above claim can be seen using the argument of [10] as follows: Let $C(\lambda)_0$ be the subspace of $C(L\bar{\mathfrak{n}}_-, M(\lambda))$ spanned by the elements

$$\widehat{J}_{a_1}(-n_1) \dots \widehat{J}_{a_p}(-n_p) \psi_{\alpha_{j_1}}(-m_1) \dots \psi_{\alpha_{j_q}}(-m_q)|\lambda\rangle \quad (a_i \in \bar{\Delta}_+ \sqcup \bar{I}, \alpha_{j_i} \in \bar{\Delta}_+).$$

Then, $d_- C(\lambda)_0 \subset C(\lambda)_0$ and

(87) $$H^\bullet_-(M(\lambda)) = H^\bullet(C(\lambda)_0), \quad H^i(C(\lambda)_0) = 0 \ (i \neq 0),$$

see [1, section 5]. Define a decreasing filtration $\{G^p C^n(\lambda)_0\}$ on $C(\lambda)_0$ by setting

$$G^p C^n(\lambda)_0 = \bigoplus_{\langle \mu - \lambda, \bar{\rho}^\vee \rangle \leq n-p} C^n(\lambda)_0^\mu$$

Then, one can check that the corresponding spectral sequence converges, and it induces a filtration on $H^0_-(M(\lambda))$ with desired properties. $\qquad\square$

Remark 6.1.3.   (1) It is not likely that $H^0_+(M(\lambda)) \cong \mathbf{M}(\gamma_{\overline{t_{-\bar{\rho}^\vee} \circ \lambda}})$ in general.
  (2) By replacing $L(0)$ with the formal degree operator defined in [1], one sees that Theorem 6.1.2 holds also for $\kappa = 0$.

6.2. **The images of the duals of Verma modules.** For $V \in Obj\mathcal{O}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$, let

$$\mathcal{HW}(V) = \{v \in V; W_i(n)v = 0 \text{ for all } i = 1, \dots, l \text{ and } n > 0\}.$$

It is naturally an $\mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$-module. Clearly, $V_{\mathrm{top}} \subset \mathcal{HW}(V)$.

Lemma 6.2.1. *Let $\bar{\lambda} \in \bar{\mathfrak{h}}^*$. Let M be an object in $\mathcal{O}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$ such that $\mathrm{ch}\,M = \mathrm{ch}\,\mathbf{M}(\gamma_{\bar{\lambda}})$ and $\mathcal{HW}(M) \cong \mathbb{C}_{\gamma_{\bar{\lambda}}}$ as $\mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$-modules. Then, $M \cong D(\mathbf{M}(\gamma_{-w_0(\bar{\lambda})}))$.*

Proof. By the assumption, $M = \bigoplus_{\Delta \in \Delta_{\bar{\lambda}} + \mathbb{Z}_{\geq 0}} M_\Delta$ and $M_{\Delta_{\bar{\lambda}}} \cong \mathbb{C}_{\gamma_{\bar{\lambda}}}$. Thus, $D(M) = \bigoplus_{\Delta \in \Delta_{\bar{\lambda}} + \mathbb{Z}_{\geq 0}} D(M)_\Delta$ and $D(M)_{\Delta_{\bar{\lambda}}} \cong \mathbb{C}_{\gamma_{-w_0(\bar{\lambda})}}$ by Lemma 5.2.2. Hence, there exists a homomorphism $\phi : \mathbf{M}(\gamma_{-w_0(\bar{\lambda})}) \to D(M)$ in $\mathcal{O}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$. Consider the exact sequence $\mathbf{M}(\gamma_{-w_0(\bar{\lambda})}) \to D(M) \to \mathrm{coker}\,\phi \to 0$. It induces an exact sequence

(88) $$0 \to D(\mathrm{coker}\,\phi) \to M \to D(\mathbf{M}(\gamma_{-w_0(\bar{\lambda})})).$$

In particular, $D(\mathrm{coker}\,\phi) \subset M$. Thus, $\mathcal{HW}(D(\mathrm{coker}\,\phi)) \subset \mathcal{HW}(M) = M_{\Delta_{\bar{\lambda}}}$. This implies $\mathcal{HW}(D(\mathrm{coker}\,\phi)) = \{0\}$, for $D(\mathrm{coker}\,\phi)_{\Delta_{\bar{\lambda}}} = 0$. But, then, $D(\mathrm{coker}\,\phi) = \{0\}$. Therefore, M is a submodule of $D(\mathbf{M}(\gamma_{-w_0(\bar{\lambda})}))$. Hence, the assumption $\mathrm{ch}\,M = \mathrm{ch}\,D(\mathbf{M}(\gamma_{-w_0(\bar{\lambda})}))$ shows that $M = D(\mathbf{M}(\gamma_{-w_0(\bar{\lambda})}))$. $\qquad\square$

Theorem 6.2.2. *Let $\lambda \in \mathfrak{h}^*$ be non-critical.*
  (1) *Suppose that $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\geq 1}$ for all $\alpha \in \bar{\Delta}_+$. Then, $H^0_-(M(\lambda)^*) \cong D(\mathbf{M}(\gamma_{-w_0(\bar{\lambda})}))$.*
  (2) *Suppose that $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\geq 1}$ for all $\alpha \in \Delta^{\mathrm{re}}_+ \cap t_{\bar{\rho}^\vee}(\Delta^{\mathrm{re}}_-)$. Then, $H^0_+(M(\lambda)^*) \cong D(\mathbf{M}(\gamma_{-w_0(\overline{t_{-\bar{\rho}^\vee} \circ \lambda})}))$.*

Proof. (1) By [1, Theorem 6,8 (2)], we have

(89) $$H^i_-(M(\lambda)^*) = \{0\} \quad (i \neq 0),$$

(90) $$H^0_-(M(\lambda)^*) \cong D(\mathbf{M}(\gamma_{\bar{\lambda}})) \text{ as } L(0)\text{-modules}.$$

under the assumption of Theorem (1). In particular,

$$
(91) \qquad\qquad \operatorname{ch} H^0_-(M(\lambda)^*) = \operatorname{ch} \mathbf{M}(\gamma_{\bar\lambda}).
$$

This further implies that

$$
(92) \qquad H^0_-(M(\lambda)^*)_{\Delta_{\bar\lambda}} = \mathbb{C}|\lambda\rangle^* \cong \mathbb{C}_{\gamma_{\bar\lambda}} \text{ as } \mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))\text{-modules.}
$$

Here, $|\lambda\rangle^* = v_\lambda^* \otimes \mathbf{1}$ and $v_\lambda^* \in M(\lambda)^*$ is the vector dual to the highest weight vector $v_\lambda \in M(\lambda)$. Thus, by Lemma 6.2.1, it is sufficient to show that $\mathcal{HW}(H^0_-(M(\lambda)^*)) = \mathbb{C}|\lambda\rangle^*$.

Now the isomorphism in (90) is not of $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$-modules. However, set

$$
G^p H^0_-(M(\lambda)^*) = D(\mathbf{M}(\gamma_{\bar\lambda})/G^{-p}\mathbf{M}(\gamma_{\bar\lambda})) \quad (p \geq -1) \text{ as } L(0)\text{-modules}
$$

via the isomorphism (90). Then,

$$
H^0_-(M(\lambda)^*) = G^{-1}H^0_-(M(\lambda)^*) \supset G^0 H^0_-(M(\lambda)^*) \supset \cdots \supset G^p H^0_-(M(\lambda)^*) \supset \ldots,
$$
$$
\bigcap_p G^p H^0_-(M(\lambda)^*) = \{0\},
$$

and we see from the proof of [1, Theorem 6,8] that

$$
(93) \qquad\qquad G^p \mathcal{W}_\kappa(\bar{\mathfrak{g}}) \cdot G^q H^0_-(M(\lambda)^*) \subset G^{p+q} H^0_-(M(\lambda)^*),
$$

and that

$$
(94) \qquad\qquad \operatorname{gr}^G H^0_-(M(\lambda)^*) \cong \operatorname{gr}^G D(\mathbf{M}(\gamma_{\bar\lambda}))
$$

as modules over the polynomial ring generated by $\bar{W}_i(n)$ $(i \in \bar{I}, n > 0)$. Here, of course, $\operatorname{gr}^G H^0_-(M(\lambda)^*) = \bigoplus_p G^p H^0_-(M(\lambda)^*)/G^{p+1}H^0_-(M(\lambda)^*)$. This shows, by (80), that

$$
\{v \in \operatorname{gr}^G H^0_-(M(\lambda)^*); \bar{W}_i(n)v = 0 \quad (i \in \bar{I}, n > 0)\} = \mathbb{C}\overline{|\lambda\rangle}^*.
$$

Here, $\overline{|\lambda\rangle}^*$ is the image of $|\lambda\rangle^*$. Hence $\mathcal{HW}(H^0_-(M(\lambda)^*)) = \mathbb{C}|\lambda\rangle^*$. (1) is proved. (2) can be similarly proved by using [1, Theorem 6.8 (1)]. $\qquad\square$

6.3. The generic Verma modules. Let $\kappa \in \mathbb{C}^*$.

Theorem 6.3.1. *Suppose that $\lambda \in \mathfrak{h}^*_\kappa$ is antidominant, i.e, $\langle\lambda+\rho, \alpha^\vee\rangle \notin \{1, 2, \ldots, \}$ for all $\alpha \in \Delta^{\mathrm{re}}_+$. Then, $\mathbf{M}(\gamma_{\bar\lambda}) = \mathbf{L}(\gamma_{\bar\lambda})$.*

Proof. We have: $M(\lambda) = M(\lambda)^* = L(\lambda)$ for an antidominant λ. But then, by Theorem 6.1.2 and Theorem 6.2.2, $\mathbf{M}(\gamma_{\bar\lambda}) = D(\mathbf{M}(\gamma_{-w_0(\bar\lambda)}))$. But by Lemma 5.2.4, this only happens when $\mathbf{M}(\gamma_{\bar\lambda}) = \mathbf{L}(\gamma_{\bar\lambda})$. $\qquad\square$

Remark 6.3.2. Let $\bar{\mathfrak{g}} = \mathfrak{sl}_2(\mathbb{C})$. Then, one can apply Theorem 6.3.1 to give yet another proof of [14, Proposition 8.2 (b)].

6.4. The functor $H^0_\pm(?)$.

Definition 6.4.1.
   (1) A infinitesimal character $\gamma_{\bar\lambda}$ $(\bar\lambda \in \bar{\mathfrak{h}}^*)$ is called *non-degenerate* if $\langle\bar\lambda, \alpha^\vee\rangle \notin \mathbb{Z}$ for all $\alpha \in \bar{\Delta}$.
   (2) A weight $\Lambda \in \mathfrak{h}$ is called *non-degenerate* if $\langle\Lambda, \bar\alpha^\vee\rangle \notin \mathbb{Z}$ for all $\bar\alpha \in \bar{\Delta}$.

Remark 6.4.2.
   (1) $\Lambda \in \mathfrak{h}^*$ is non-degenerate if and only if $R^\Lambda \cap \bar{\Delta} = \emptyset$. Therefore, Λ is non-degenerate if and only if $w \circ \Lambda$ is non-egenerate for all $w \in W^\Lambda$.

(2) The integral Weyl group W^Λ can be an infinite group even when $\Lambda \in \mathfrak{h}^*$ is non-degenerate. Indeed, $W^\Lambda \cong W$ for a principal admissible weight Λ, see [15, 16].

Theorem 6.4.3 ([1, Theorem 8.3]). *Let $\kappa \in \mathbb{C}^*$.*

(1) *Suppose $\Lambda \in \mathfrak{h}_\kappa^*$ is non-degenerate. Then, $H_-^i(V) = 0$ $(i \neq 0)$ for all $V \in \mathcal{O}_\kappa^{[\Lambda]}$.*

(2) *Suppose $\Lambda \in \mathfrak{h}_\kappa^*$ satisfies the following condition.*

$$(95) \qquad\qquad R^\Lambda \cap \Delta_+^{\mathrm{re}} \cap t_{\bar{\rho}^\vee}(\Delta_-^{\mathrm{re}}) = \emptyset.$$

Then, $H_+^i(V) = 0$ $(i \neq 0)$ for all $V \in \mathcal{O}_\kappa^{[\Lambda]}$.

Remark 6.4.4.

(1) The condition (95) is equivalent to

$$(96) \qquad \langle \Lambda, \alpha^\vee \rangle \notin \mathbb{Z} \quad \text{for all } \alpha \in \{-\bar{\alpha} + n\delta; \bar{\alpha} \in \bar{\Delta}_+, 1 \leq n \leq \mathrm{ht}\, \bar{\alpha}\}.$$

(2) If $\Lambda \in \mathfrak{h}^*$ satisfies (95), then $t_{-\bar{\rho}^\vee} \circ \Lambda$ is non-degenerate.

Corollary 6.4.5. *Let $\kappa \in \mathbb{C}^*$.*

(1) *Suppose $\Lambda \in \mathfrak{h}_\kappa^*$ is non-degenerate. Then, the correspondence $V \rightsquigarrow H_-^0(V)$ defines an exact functor from $\mathcal{O}_\kappa^{[\Lambda]}$ to $\mathcal{O}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$.*

(2) *Suppose $\Lambda \in \mathfrak{h}_\kappa^*$ satisfies (95). Then, the correspondence $V \rightsquigarrow H_+^0(V)$ defines an exact functor from $\mathcal{O}_\kappa^{[\Lambda]}$ to $\mathcal{O}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$.*

6.5. The images of simple modules ($(-)$-case).

Theorem 6.5.1. *Suppose that $\lambda \in \mathfrak{h}^*$ is non-degenerate and non-critical. Then, $H_-^0(L(\lambda)) \cong \mathbf{L}(\gamma_{\bar{\lambda}})$.*

Proof. By Corollary 6.4.5, Theorem 6.1.2 and Theorem 6.2.2, the exact sequences $M(\lambda) \to L(\lambda) \to 0$ and $0 \to L(\lambda) \to M(\lambda)^*$ in $\mathcal{O}_\kappa^{[\lambda]}$ give rise to the exact sequences

$$(97) \qquad\qquad \mathbf{M}(\gamma_{\bar{\lambda}}) \to H_-^0(L(\lambda)) \to 0,$$

$$(98) \qquad\qquad 0 \to H_-^0(L(\lambda)) \to D(\mathbf{M}(\gamma_{-w_0(\bar{\lambda})})).$$

Thus, by Lemma 5.2.4, $H_-^0(L(\lambda))$ must be either $\{0\}$ or isomorphic to $\mathbf{L}(\gamma_{\bar{\lambda}})$. But [1, Proposition 7.6] shows that $H_-^0(L(\lambda)) \neq \{0\}$. $\square$

Theorem 6.5.1 in particular proves the irreducibility conjecture of Frenkel, Kac and Wakimoto, see [11, Conjecture $3.4_-(a)$]. One can also apply Theorem 6.4.3 and Theorem 6.5.1 to non-principal admissible weights to prove the conjecture [11, Proposition 3.4 (c)].

6.6. The characters. Let

$$(99) \qquad \mathcal{C}_\pm^\kappa = \{\Lambda \in \mathfrak{h}_\kappa^*; \langle \Lambda + \rho, \alpha^\vee \rangle \notin \{\mp 1, , \mp 2, \ldots, \} \text{ for all } \alpha \in \Delta_+^{\mathrm{re}}\},$$

$$(100) \qquad \mathcal{C}_{\pm,\mathrm{nondeg}}^\kappa = \{\Lambda \in \mathcal{C}_\pm^\kappa; \Lambda \text{ is non-degenerate}\} \subset \mathcal{C}_\pm^\kappa.$$

For $\Lambda \in \mathfrak{h}^*$, let $\geq_\Lambda$ denote the Bruhat ordering in W^Λ and let ℓ_Λ denote the length function on W^Λ. Let $P_{w,y}^\Lambda$ $(w, y \in W^\Lambda)$ be the corresponding Kazhdan-Lusztig polynomial and $Q_{w,y}^\Lambda$ the inverse Kazhdan-Lusztig polynomial. Let $W_0^\Lambda = \langle s_\alpha; \langle \Lambda + \rho, \alpha^\vee \rangle = 0 \rangle \subset W^\Lambda$.

The following theorem describes the normalized characters of all irreducible representations of $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$ of non-generate highest weight $\gamma_{\bar{\lambda}}$.

Theorem 6.6.1. *Let $\kappa \in \mathbb{C}^*$.*

(1) *Let $\Lambda \in \mathcal{C}^\kappa_{+,\mathrm{nondeg}}$. Then, for any $w \in W^\Lambda$ which is the longest element of wW_0^Λ we have*

$$\mathrm{ch}(\mathbf{L}(\gamma_{\overline{w\circ\Lambda}})) = \frac{1}{\eta(\tau)^{\mathrm{rank}\,\bar{\mathfrak{g}}}} \sum_{W^\Lambda \ni y \geq_\Lambda w} (-1)^{\ell_\Lambda(y)-\ell_\Lambda(w)} Q^\Lambda_{w,y}(1) q^{\frac{|y(\Lambda+\rho)|^2}{2\kappa}}.$$

(2) *Let $\Lambda \in \mathcal{C}^\kappa_{-,\mathrm{nondeg}}$. Then, for any $w \in W^\Lambda$ which is the shortest element of wW_0^Λ we have*

$$\mathrm{ch}(\mathbf{L}(\gamma_{\overline{w\circ\Lambda}})) = \frac{1}{\eta(\tau)^{\mathrm{rank}\,\bar{\mathfrak{g}}}} \sum_{W^\Lambda \ni y \leq_\Lambda w} (-1)^{\ell_\Lambda(y)-\ell_\Lambda(w)} P^\Lambda_{w,y}(1) q^{\frac{|y(\Lambda+\rho)|^2}{2\kappa}}.$$

Proof. By Corollary 6.4.5 and Theorem 6.5.1, Theorem follows directly from the character formula [18, Theorem 1.1] in $\mathcal{O}_\kappa$. □

6.7. The images of simple modules ((+)-case).

Theorem 6.7.1. *Suppose that $\lambda \in \mathfrak{h}^*$ is non-critical and satisfies the condition (95). Then, $H^0_+(L(\lambda)) \cong \mathbf{L}(\gamma_{\overline{t_{-\bar{\rho}^\vee}\circ\lambda}})$.*

Proof. By Theorem 6.2.2 (2) and Corollary 6.4.5 (2), the condition on λ implies that $H^0_+(L(\lambda))$ is a submodule of $D(\mathbf{M}(\gamma_{-w_0(\overline{t_{-\bar{\rho}^\vee}\circ\lambda})}))$. Therefore, it is sufficient to show that $\mathrm{ch}\, H^0_+(L(\lambda)) = \mathrm{ch}\,\mathbf{L}(\gamma_{\overline{t_{-\bar{\rho}^\vee}\circ\lambda}})$. But this follows from [1, Remark 3.5], Remark 6.4.4 (2), Corollary 6.4.5 and Theorem 6.5.1. □

6.8. The simple vertex operator algebra. Let

$$\mathrm{vac} = \overline{t_{-\bar{\rho}^\vee}\circ(\kappa-h^\vee)\Lambda_0} = -\kappa\bar{\rho}^\vee \in \bar{\mathfrak{h}}^*.$$

Proposition 6.8.1. *There exist a unique surjection $\mathbf{M}(\gamma_{\mathrm{vac}}) \twoheadrightarrow \mathcal{W}_\kappa(\bar{\mathfrak{g}})$ of $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$-modules that sends $|\gamma_{\mathrm{vac}}\rangle$ to $|0\rangle$.*

Proof. By Theorem 6.1.2, it is sufficient to show that $\mathbb{C}|0\rangle \cong \mathbb{C}|t_{-\bar{\rho}^\vee}\circ(\kappa-h^\vee)\Lambda_0\rangle$ as $\mathcal{A}(\mathcal{W}_\kappa(\bar{\mathfrak{g}}))$-modules. The proof of this assertion is the same as the proof of Lemma 6.1.1 (2). □

Proposition 6.8.1 implies the following result:

Theorem 6.8.2. *For any $\kappa \in \mathbb{C}^*$, $\mathbf{L}(\gamma_{\mathrm{vac}})$ is the unique simple quotient of $\mathcal{W}_\kappa(\bar{\mathfrak{g}})$. In particular, $\mathbf{L}(\gamma_{\mathrm{vac}})$ carries a vertex operator algebra structure.*

Of course the character of $\mathbf{L}(\gamma_{\mathrm{vac}})$ is expressed by Theorem 6.6.1 when $\mathrm{vac} \in \bar{\mathfrak{h}}^*$ is non-degenerate, i.e, when $\kappa(\bar{\rho}^\vee, \alpha^\vee) \notin \mathbb{Z}$ for all $\alpha \in \bar{\Delta}$.

6.9. The conjecture.

Conjecture 1.

(1) For any $\kappa \in \mathbb{C}$ and any $V \in Obj\mathcal{O}_\kappa$, $H^i_-(V) = 0$ ($i \neq 0$).
(2) For any non-critical weight λ,

$$H^0_-(L(\lambda)) \cong \begin{cases} \mathbf{L}(\gamma_{\bar{\lambda}}) & \text{if } \langle\lambda+\rho, \bar{\alpha}^\vee\rangle \notin \{1, 2, \dots, \} \text{ for all } \bar{\alpha} \in \bar{\Delta}_+, \\ 0 & \text{otherwise.} \end{cases}$$

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